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Shapes of Growing Droplets—A Model of Escape from a Metastable Phase

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Nucleation from a metastable state is studied for an Ising ferromagnet with nearest and next nearest neighbor interaction and at very low temperatures. The typical escape path is shown to follow a sequence of configurations with a growing droplet of stable phase whose shape is determined by dynamical considerations and differs significantly from the equilibrium shape corresponding to the instantaneous volume.

KEY WORDS: Stochastic dynamics; Ising model; next nearest neighbor interaction; metastability; crystal growth; first excursion.

1. INTRODUCTION

Relaxation to equilibrium of a system close to a first-order transition is a problem whose rigorous treatment has recently been receiving a lot of attention.[14,15,11,12,10] In particular, it is of interest to grasp the escape pattern of a system relaxing from a metastable starting configuration toward a stable equilibrium state. The escape is through the formation of small droplets or crystals of stable phase that are stabilized once they reach a certain critical size. Here, we are interested in the shape of such a crystal during the process of growth. It seems that the mechanism of growth depends on the size of the crystal. While for large supercritical crystals one has to take into account the transport of matter and heat around the crystal, the growth of tiny subcritical crystals should be governed in a more direct way by the instantaneous microscopic dynamics.

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Our aim in the present paper is to discuss the shape of a growing crystal modeled by the Ising model. Namely, we consider the Ising model with ferromagnetic nearest neighbor and next nearest neighbor interactions in the presence of a small positive external field. Starting from the configuration $-\underline{1}$, minus spins at all lattice sites in a fixed finite volume under periodic boundary conditions, we study the relaxation pattern of the stochastic process yielded by a standard Glauber dynamics. In particular, we are interested in the typical configurations during the first excursion from the configuration $-\underline{1}$ to the configuration $+\underline{1}$ with all sites occupied by plus spins. We present a detailed description of the escape pattern in the asymptotic region of vanishing temperatures. It turns out that in this asymptotics (and in a finite volume) one can consider a single droplet of plus spins, playing the role of the crystal, that grows in a very particular manner to the critical droplet.

At first sight one could suppose that growth is through crystals that minimize the surface tension under fixed instantaneous volume.[17] The shape of such crystals is the equilibrium shape yielded by the Wulff construction.[16,13,4] For our model at low temperatures the Wulff construction can be easily shown to lead to a shape that closely follows an octagon with coordinate and oblique sides proportional to the nearest neighbor and next nearest neighbor interactions, respectively. The main result of the present paper (Theorem 3) asserts that the typical growth of subcritical crystals is through a sequence of particular shapes that significantly differ from the equilibrium Wulff octagons. This is in agreement with a similar result[8] concerning the Ising model with anisotropic nearest neighbor interaction. Notice that the fact that we are considering the asymptotics of vanishing temperature with fixed (small) external field is technically crucial for our proofs. It would be interesting (and difficult) to extend this type of result to the region of vanishing external field under a fixed (small) temperature or to the region where both external field and temperature vanish in such a way that their ratio is fixed. The escape time and asymptotics of the metastable state in the latter region have been recently discussed[15] in the case of the nearest neighbor d-dimensional Ising model.

We believe that a difference between dynamical and equilibrium shapes would occur already for the simplest nearest neighbor Ising model. But while it would reveal itself only as a higher-order effect at low temperatures, for the model with an additional next nearest neighbor interaction discussed here it appears already in the first order of the low-temperature asymptotics. The main effect is thus captured in a situation that is sufficiently simple to be studied in a rigorous way.

The main principle according to which typical growth patterns are determined can be summarized, in a nontechnical and simplified manner, as follows. The instantaneous shape of a crystal determines its (microscopic) energy and the growth can be thought of as a path through this energy landscape. Let us first discuss the behavior in the neighborhood of a configuration Q at which the energy attains a local minimum. Considering the standard Glauber dynamics, we can pose the following question: what is the escape path from the "basin of attraction" $\mathscr{B}(Q)$ of the local minimum Q? The answer, in the general context of reversible Markov chains with transition probabilities exponentially decreasing with inverse temperature β, is not surprising. Namely, general arguments based on reversibility lead to the conclusions that:

(i) The most probable way out of $\mathscr{B}(Q)$ is through a minimal saddle point S on the boundary $\partial\mathscr{B}(Q)$ of $\mathscr{B}(Q)$ [the configuration of minimal energy among those that are outside $\mathscr{B}(Q)$ but are connected by a single spin flip with a configuration in $\mathscr{B}(Q)$].

(ii) The typical path is through a sequence of configurations "against the drift," namely, the time reversal $R\omega$ of the path ω starting at S and descending to Q.

The task of describing the global escape from $-\underline{1}$ to $+\underline{1}$ is complicated by the fact that one is moving through an energy landscape with many local minima. The above observation about the escape from the basin of attraction of a given local minimum has to be used for different minima and combined into a global picture. Local minima are yielded by certain particulary simple shapes (general octagons in our case) with all faces completely filled up (no microscopic holes or other perturbations on facets). The probability that, starting from a given local minimum Q_1 the crystal grows (or shrinks) to a neighboring local minimum Q_2 [microscopically by adding (or erasing) one microscopic layer to (or from) one or several faces of Q_1] is determined by the height $H(S)-H(Q_1)$ of the barrier between them. We have here in mind any configuration S at which the energy on a path from Q_1 to Q_2 reaches its maximum, however, with the path chosen to minimize it. In terms of this energy of the barrier, the considered probability is proportional to $\exp[-\beta\{H(S)-H(Q_1)\}]$. Having this "microscopic building block," our task is to determine the probability of reaching the global saddle point (critical nucleus) of relative energy Γ (with respect to the starting metastable configuration $-\underline{1}$). Multiplying naively the probabilities corresponding to passages through subsequent local saddle points would yield a gross underestimation of this probability expected to be of the order $e^{-\beta\Gamma}$ (the typical time of escape from the metastable state is expected to be $e^{\beta\Gamma}$).

To get a more accurate evaluation we have to take into account also the following circumstance. A crystal at a local energetic minimum is likely to "stay" in its basin of attraction for the time of the order $\exp[\beta\{H(\bar{S})-H(Q_1)\}]$, where $\bar{S}$ is the lowest local saddle point through which the crystal can escape from the local minimum Q_1 (not necessarily in "the direction of" Q_2). Being thus slightly more patient and allowing the crystal to pass through the (higher) local saddle point S (on the way to Q_2) starting at any randomly chosen time before $\exp[\beta\{H(\bar{S})-H(Q_1)\}]$, we get the probability of the order

$$e^{\beta\{H(\bar{S})-H(Q_1)\}}e^{-\beta\{H(S)-H(Q_1)\}}=e^{\beta\{H(S)-H(\bar{S})\}}$$

Combining now these contributions, we have a chance to get a correct estimate of the order $e^{-\beta\Gamma}$ for a global path once it meets the obvious local condition. Namely, for each visited local energetic minimum, *the path* (or rather a class of paths defined by a sequence of visited local minima) *has to enter its basin of attraction through the lowest possible local saddle point.* This is a severe restriction on possible paths (all others being much less probable) and using it we get the most probable growth patterns mentioned above.

Unfortunately, the technical details to achieve the above strategy are rather complex.

The paper is organized in the following way. In Section 2 we introduce the model, the dynamics, and the notation concerning octagonal shapes that yield local minima of the considered interaction (Lemma 2.1). Then we summarize our results in Theorems 1 and 2 (asymptotics of the hitting time to the configuration $+\underline{1}$) and Theorem 3 (describing the sequence of droplet shapes on the escape path).

In Section 3 we begin the discussion of "the movement in the energy landscape" by a detailed investigation of passages between neighboring (octagonal) local minima. An important role is played by a characterization of different basins of attraction and saddle points on their boundaries. A closely linked fact is the existence of two different important time scales in the problem. The shorter one is a typical time needed for a passage from a particular octagonal local minimum to a close octagon with identical circumscribed rectangle. The height of the corresponding saddle point is proportional to the value of the next nearest neighbor coupling. Only when passing to an octagon with larger circumscribed rectangle does one have to overcome a saddle point whose height is proportional to the nearest neighbor coupling and the typical passage time is correspondingly longer (We are supposing that the nearest neighbor coupling is stronger than the next nearest neighbor one.) In terms of this longer time scale we observe a growing octagonal shape whose oblique sides are "breathing" around "equilibrium positions" when observed on a shorter time scale.

The global saddle point, a configuration minimizing the maximal energies on paths from $-\underline{1}$ to $+\underline{1}$, is discussed in Section 4. For its investigation it is useful to introduce a "global basin of attraction of the configuration $-\underline{1}$," respectively its subset $\mathscr{A}$ characterized by the fact that starting from a configuration in $\mathscr{A}$, a typical path first hits the configuration $-\underline{1}$ before reaching the global minimum $+\underline{1}$. The goal is to choose the set $\mathscr{A}$ small enough to satisfy this condition but large enough to make the minimum on its boundary coincide with the global saddle point.

The results of Section 3 and 4 are than merged in Section 5 into the proofs of Theorems 1–3. The basic estimate in getting a typical escape time is the lower bound on the probability to reach a global saddle point. Here we use the stategy suggested above, taking into account "resistance times" characterized by the minimal saddle point on the boundary of the local basin of attraction of an octagonal local minimum configuration. The resulting local condition determines the optimal escape path and is responsible for a particular dynamically optimal sequence of shapes mentioned above.

The results of the present paper, as well as those from ref. 8, were announced in ref. 9.

2. SETTING AND RESULTS

We will consider a *discrete-time Metropolis dynamics* for a totally ferromagnetic two-dimensional *Ising model* with *nearest neighbors* and *next neighbor interaction.*

The choice of a discrete- instead of a continuous-time evolution is made only for the sake of simplicity of the exposition. It will appear clear to the reader that, with some minor changes, all our results can be extended to the continuous case. Our dynamics will be given by a Markov chain whose space of states is $\Gamma = \{-1, 1\}^{\Lambda}$ where Λ is a two-dimensional torus, namely, Λ is the square $\{1, ..., M\}^2$ with periodic boundary conditions.[3]

A configuration σ is a function

$$\sigma: \Lambda \to \{-1, 1\} \tag{2.1}$$

i.e., $\sigma \in \Gamma = \{-1, 1\}^{\Lambda}$. The value $\sigma(x)$ is the *spin* at the site x. The *energy* of a configuration σ is

$$H(\sigma) = -\frac{J}{2} \sum_{\langle x, y \rangle \subset \Lambda} \sigma(x)\,\sigma(y) - \frac{K}{2} \sum_{\langle\langle x, y \rangle\rangle \subset \Lambda} \sigma(x)\,\sigma(y) - \frac{h}{2} \sum_{x \in \Lambda} \sigma(x) \tag{2.2}$$

[3] For instance, $x = (M+1, b)$ is, for every $b \leqslant M$, identified with $x' = (1, b)$ and so on.

where we suppose that K, $\tilde{J}$, $h > 0$. Here $\langle x, y \rangle$ denotes a pair of nearest neighbors in Λ: $|x-y| = 1$ and $\langle\langle x, y \rangle\rangle$ denotes a pair of next nearest neighbors in Λ: $|x-y| = \sqrt{2}$. Our dynamics is defined with the help of the following *updating rule*:

Given the configuration σ at time t, we first choose at random (with uniform probability) a site $x \in \Lambda$. Then we flip the spin at the site x with probability

$$\exp\{-\beta[\Delta_x H(\sigma)]^+\} \tag{2.3}$$

where

$$\Delta_x H(\sigma) = H(\sigma^{(x)}) - H(\sigma) \tag{2.4}$$

with

$$\sigma^{(x)}(y) = \begin{cases} \sigma(y) & \text{whenever} \quad y \neq x \\ -\sigma(x) & \text{whenever} \quad y = x \end{cases} \tag{2.5}$$

Here, for every $c \in \mathbb{R}$ we denote $[c]^+ = \min(c, 0)$; β is the inverse temperature.

The *transition probabilities* are then given by

$$P(\sigma \to \eta) = \begin{cases} (1/|\Lambda|) \exp\{-\beta[\Delta_x H(\sigma)]^+\} & \text{if} \quad \eta = \sigma^{(x)} \text{ for some } x \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

The *space of the trajectories* of our process is

$$\Omega = \Gamma^{\mathbb{N}} \equiv (\{-1, 1\}^{\Lambda})^{\mathbb{N}} \tag{2.7}$$

An element in Ω is denoted by ω; it is a function

$$\omega: \quad \mathbb{N} \to \Gamma$$

If

$$\omega = \sigma_0, \sigma_1, \ldots, \sigma_t, \ldots$$

we set

$$\omega_t \equiv \omega\restriction_t = \sigma_t$$

The dynamics is *reversible* with regard to the Gibbs measure in Λ in the sense that

$$P(\sigma \to \eta)\, e^{-\beta H(\sigma)} \equiv P(\eta \to \sigma)\, e^{-\beta H(\eta)} \tag{2.8}$$

We will discuss the behavior at very low temperatures. Thus it is natural to describe configurations in terms of contours. Namely, for every $\sigma \in \Gamma$ consider the union $C(\sigma)$ of all closed unit squares centered at lattice sites x for which $\sigma(x) = +1$. The boundary ∂C of C is a polygon with vertices on the dual lattice $\mathbb{Z}^2 + (1/2, 1/2)$ such that in any vertex of the dual lattice an even number (zero, two, or four) of unit segments belonging to this polygon meet. Any connected component γ of the boundary ∂C is called a *contour* of the configuration σ.

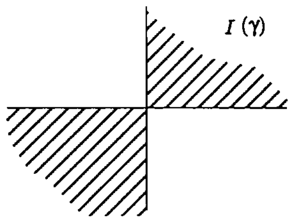
It is easy to see that if a configuration σ has the boundary $\partial C(\sigma)$ consisting of a single contour γ, the energy of σ is

$$H(\sigma) - H(-\underline{1}) = J\,|\gamma| - K\,|A(\gamma)| - h\,|I(\gamma)| \tag{2.9}$$

Here

$$J = \tilde{J} + 2K \tag{2.10}$$

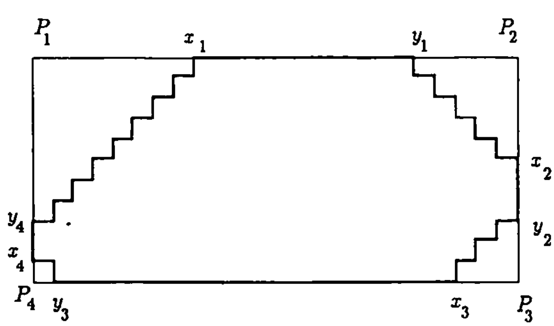
$|\gamma|$ is the length of γ, and $|I(\gamma)|$ is the cardinality (area) of the interior $I(\gamma) \equiv C(\sigma)$. Finally, $|A(\gamma)|$ is the number of corners (right angles) of γ. Notice that in the situation like that on Scheme 2.1 we count four corners.



Scheme 2.1

A contour γ is said to be *isolated* if it lies at a distance at least $\sqrt{2}$ from other contours.

A relevant role will be played by a particular class of contours that we call octagons. An *octagon* is a closed contour inscribed in a rectangle R with edges parallel to the lattice axes. Call P_1, P_2, P_3, P_4 the vertices of R. The octagon contains four straight edges with extremes x_i, y_i, $i = 1,..., 4$: $(x_1, y_1) \subseteq P_1P_2$, $(x_2, y_2) \subseteq P_2P_3$, $(x_3, y_3) \subseteq P_3P_4$, and $(x_4, y_4) \subseteq P_4P_1$, called *coordinate edges*, and four *oblique edges* that have a local staircase structure with extremes y_1x_2, y_2x_3, y_3x_4, y_4x_1 (see Scheme 2.2).



Scheme 2.2

When referring to a *stable octagon* we suppose that $|x_i - y_i| \geq 2$, $|y_i - x_{i+1}| \geq \sqrt{2}$, $i = 1,..., 4$. When discussing a particular octagon Q, it will be useful to introduce

$$L_i = |x_i - y_i|, \qquad i = 1,..., 4 \tag{2.11}$$

for the lengths of its coordinate edges;

$$l_i = 1 + \frac{1}{\sqrt{2}} |y_i - x_{i+1}|, \qquad i = 1,..., 4 \quad (x_5 \equiv x_1) \tag{2.12}$$

for the lengths of its oblique edges;

$$\begin{aligned} D_1 &= \overline{P_1 P_2} = L_1 + l_4 - 1 + l_1 - 1 = \overline{P_4 P_3} = L_3 + l_3 - 1 + l_2 - 1 \\ D_2 &= \overline{P_2 P_3} = L_2 + l_1 - 1 + l_2 - 1 = \overline{P_4 P_1} = L_4 + l_3 - 1 + l_4 - 1 \end{aligned} \tag{2.13}$$

for the lengths of the sides of its *rectangular envelope* $R(Q) = P_1 P_2 P_3 P_4$; and, finally,

$$\begin{aligned} d_1 &= L_1 + L_4 + 2(l_4 - 1) = L_2 + L_3 + 2(l_2 - 1) \\ d_2 &= L_1 + L_2 + 2(l_1 - 1) = L_4 + L_3 + 2(l_3 - 1) \end{aligned} \tag{2.14}$$

for the distances (in units of $1/\sqrt{2}$) between pairs of opposite parallel oblique edges. We will use

$$Q(D_1, D_2, l_1, l_2, l_3, l_4) \tag{2.15}$$

to denote the corresponding octagon. Another way of characterize it is by specifying, say, $d_1, d_2, L_1, L_2, L_3, L_4$.

The previous definitions made no reference to the *location of the octagon*. Sometimes in the sequel we will consider a canonical location. We say that an octagon Q is *centered* if the upper left corner x_1 of Q is the point $(-1/2, 1/2)$ of the dual lattice (namely, the uppermost left + spin, the first in lexiographic order, is the origin). Sometimes when not specifying the location of an octagon Q we tacitly assume it to be centered.

We will often use the same symbol Q also to denote the set of all spin configurations σ giving rise to a unique closed contour (full of plusses) consisting, up to a translation on the torus, of the octagon Q; sometimes we write $\sigma \in Q$. We set

$$H(Q) \equiv H(\sigma) \qquad \text{for} \quad \sigma \in Q$$

Clearly, for $Q = Q(D_1, D_2, (l_i)_{i=1,\dots,4})$ we have

$$H(Q) = 2J(D_1 + D_2) - hD_1 D_2 - \sum_{j=1}^{4} \left[K(2l_j - 1) - \tfrac{1}{2}hl_j(l_j - 1)\right] \quad (2.16)$$

Let us define

$$l^* = \left[\frac{2K}{h}\right] + 1, \qquad \text{i.e.,} \quad h(l^* - 1) < 2K < hl^*$$

$$D^* = \left[\frac{2J}{h}\right] + 1, \qquad \text{i.e.,} \quad h(D^* - 1) < 2J < hD^* \quad (2.17)$$

$$L^* = D^* - 2(l^* - 1) = \left[\frac{2J}{h}\right] + 1 - 2\left[\frac{2K}{h}\right]$$

Here $[x]$ denotes the integer part of a real number x. We suppose also that $2K/h$ and $2J/h$ are not integers. Notice that if η is defined as

$$\eta = \frac{2K}{h} - (l^* - 1) = \frac{2K}{h} - \left[\frac{2K}{h}\right] \quad (2.18)$$

then $2\tilde{J}/h + 2\eta$ is not an integer and

$$L^* = \left[\frac{2\tilde{J}}{h} + 2\eta\right] + 1 \quad (2.19)$$

We always suppose in the present paper that $K \leqslant J/10$ and $0 < 7h < K$. These are essentially technical assumptions that make the proofs simpler. We did not attempt to optimalize them and it might turn out that weaker conditions would also work.

We define a *standard octagon* as an octagon with

$$\min_{i=1,\dots,4} L_i \geqslant l^* \qquad \text{and} \qquad l_j = l^*, \qquad j = 1,\dots, 4$$

In this case we simply write

$$Q(D_1, D_2) \equiv Q(D_1, D_2, l^*, l^*, l^*, l^*) \quad (2.20)$$

We call an octagon *regular* when

$$L_1 = L_2 = L_3 = L_4 = l_1 = l_2 = l_3 = l_4 = l$$

and denote it by $Q(l)$. We will only be interested in regular octagons with $l \leqslant l^*$. For configurations $\sigma \in Q(D_1, D_2, l_1, l_2, l_3, l_4)$ we often write

$$D_j = D_j(\sigma), \qquad j = 1, 2$$
$$l_j = l_j(\sigma), \qquad j = 1, \ldots, 4$$

The assumption about the smallness of h made above ensures, in particular, that for every spin flip the energy changes at least by h. (See Lemma 3.1 in Section 3.)

The importance of octagons stems from the fact that they yield the set of all local minima of energy.

Lemma 2.1. Let σ be a configuration whose energy increases under every spin flip. Then the set $\{x, \sigma(x) = +1\}$ is a union of isolated stable octagons.

The proof of this lemma will be given in Section 3.

Given any set of configurations $\Delta \subset \Gamma$ we define the *first hitting time* to Δ as

$$\tau_\Delta = \inf\{t \geqslant 0 \colon \sigma_t \in \Delta\} \tag{2.21}$$

Sometimes we use the notation $P_\eta(\cdot)$ to denote the probability distribution over the process starting at $t = 0$ from a configuration η.

As we mentioned in the Introduction, we shall discuss the asymptotic behavior, in the limit $\beta \to \infty$, of typical paths of first escape from $-\underline{1}$ to $+\underline{1}$. We refer to refs. 11, 12, 14, 15, 8, and 9 for a general introduction to this problem. Also in the present case, local minima will play a crucial role in the description of the transition from the metastable situation to the stable one. It will happen, similarly to the case of the nearest neighbor Ising model, that small octagons have a tendency to shrink, whereas large ones tend to grow. Again, the dynamical mechanism responsible for this behavior relies on the competition between the creation of a suitable stable protuberance and the erosion of an edge. The main difference here with respect to the nearest neighbor Ising model comes from the presence of the oblique edges in stable isolated clusters—the octagons. The new phenomenon, namely the growth or shrinking in the oblique direction, is still governed by a competition between the creation of a certain protuberance and the erosion of an edge. However, three new features should be mentioned: (i) The corresponding time scales are different with respect to the appearing in the growth-shrinking mechanism in the coordinate directions;

(ii) for sufficiently large octagons, a sort of "equilibrium" in the oblique direction is established; and (iii) for "relevant" octagons, the global tendency to shrink or grow—to disappear or to invade the whole volume—is governed by the size of the "circumscribed rectangle."

All these mechanisms will be described in the next sections starting from the "elementary" events, taking place at microscopic level and involving a single spin flip, then analyzing the transitions between neighboring minima, and finally considering the global transition between $-\underline{1}$ and $+\underline{1}$. A very important role will be played by standard octagons that can be considered as the most stable ones among the octagons with a given, sufficiently large, circumscribed rectangle (see Section 3). The *basin of attraction* of a local minimum will be defined in Section 3 in a natural way and neighboring minima, namely the ones with a nonempty intersection of the boundaries of their basins of attraction, will be considered.

To define a *saddle point* between two neighboring local minima, say Q and Q', we consider the minimax

$$\min_{\omega: Q \to Q'} \max_{\sigma \in \omega} H(\sigma) \tag{2.22}$$

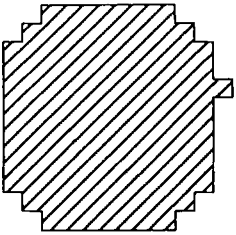
(where we use $\omega: \sigma \to \tau$ to denote a generic path by subsequent spin flips starting from σ and ending in τ). A saddle point is any configuration $\bar{\sigma}$ for which the above minimax is attained. Namely, considering any $\omega_0: Q \to Q'$ for which the minimum in (2.22) is reached, any configuration $\bar{\sigma}$ for which $\max_{\sigma \in \omega_0} H(\sigma)$ is attained is a saddle point.

It will become clear that, for sufficiently large sizes, it is interesting to consider not only the jumps between neighboring generic octagons (with related basins of attraction and saddle configurations) but also the transitions between standard octagons $Q(D_1, D_2)$ with corresponding "domains of attraction" (see Section 3) that group together all the basins of attraction of octagons inscribed in the same rectangle R as $Q(D_1, D_2)$. These transitions at the "intermediate level" (corresponding to the exit from domains of attraction instead of basins of attraction) involve an intermediate time scale between the one referring to the transition between neighboring generic octagons and the global one referring to the transition between $-\underline{1}$ and $+\underline{1}$.

A *global saddle point* is any configuration $\bar{\sigma}$ for which the minimax

$$\min_{\omega: -\underline{1} \to +\underline{1}} \max_{\sigma \in \omega} H(\sigma) \tag{2.23}$$

is attained. Let $\mathscr{P}$ be the set of all configurations obtained from a standard octagon $Q(D^*, D^* - 1)$ or $Q(D^* - 1, D^*)$ [D^* has been defined in Eq. (2.17)] by attaching to one of its longer coordinate edges a unit-square protuberance (see Scheme 2.3). We will see that $\mathscr{P}$ coincides with the set of all global saddle points.



Scheme 2.3

For any $\bar{\sigma} \in \mathscr{P}$ one has

$$H(\bar{\sigma}) - H(-\underline{1}) \equiv E^* = H(Q(D^*, D^* - 1)) + 2\tilde{J} - h$$
$$= H(Q(D^*, D^*)) + h(D^* - 1) - 4K \qquad (2.24)$$

for the "height" of the global saddle point.

We shall prove that the first excursion from $-\underline{1}$ to $+\underline{1}$ typically passes through a configuration from $\mathscr{P}$ and the time needed for this to happen is of the order $\exp(\beta E^*)$.

To present this statement in a formal way, we use $\bar{\tau}_{-\underline{1}}$ to denote the last instant in which $\sigma_t = -\underline{1}$ before $\tau_{+\underline{1}}$,

$$\bar{\tau}_{-\underline{1}} = \max\{t < \tau_{+\underline{1}} : \sigma_t = -\underline{1}\} \qquad (2.25)$$

and introduce

$$\bar{\tau}_{\mathscr{P}} = \min\{t > \bar{\tau}_{-\underline{1}} : \sigma_t \in \mathscr{P}\} \qquad (2.26)$$

Theorem 1:

$$\lim_{\beta \to \infty} P_{-\underline{1}}(\bar{\tau}_{\mathscr{P}} < \tau_{+\underline{1}}) = 1 \qquad (2.27)$$

Theorem 2:

$$\lim_{\beta \to \infty} P_{-\underline{1}}(\exp[\beta(E^* - \varepsilon)] < \tau_{+\underline{1}} < \exp[\beta(E^* + \varepsilon)]) = 1 \qquad (2.28)$$

for every $\varepsilon > 0$.

In addition, we get much more detailed information about a typical path followed by our process σ_t during its first excursion from $-\underline{1}$ to $+\underline{1}$, or, in other words, between the moments $\bar{\tau}_{-\underline{1}}$ and $\tau_{+\underline{1}}$. Theorem 3 below states, roughly speaking, that with high probability for large β, the path σ_t sticks to a certain *tube* of trajectories. A precise definition involves a lot of other preliminary definitions and notions and we will ba able to present it only at the end of Section 5. There, we will introduce the concept of an *ε-typical path* that will be determined in terms of its geometrical properties, but also with a specified time of passing through certain configurations. For the moment we only say that, roughly speaking, the typical trajectories during the first excursion begin by following a sequence of almost regular octagons up to an edge l^*; after that the oblique edges stay almost constant at the value l^* while the coordinate edges grow, keeping the rectangular envelope almost squared, from the value l^* up to a value L^* corresponding to the critical nucleus. This first part of the first excursion can be viewed as a nucleation phenomenon and it involves, in average, "ascending" transitions with growing energy. Finally, the oblique edges stay further almost constant at the value l^* whereas the coordinate edges continue to grow, with larger fluctuations, still preserving, however, the average squared shape of the rectangular envelope until the whole volume is invaded by plusses. This "supercritical growth" is, in average, a descent in energy.

Theorem 3. For every $\varepsilon > 0$ one has

$$\lim_{\beta \to \infty} P_{-\underline{1}}(\{\sigma_t\}_{t \in [\bar{\tau}_{-\underline{1}}, \tau_{+\underline{1}}]} \text{ is an } \varepsilon\text{-typical path}) = 1 \qquad (2.29)$$

3. PASSAGE BETWEEN NEIGHBORING LOCAL MINIMA

To be able to discuss in detail the growth or shrinking of a droplet we first introduce some "elementary events," namely, certain particular spin flips.

Notice first that the energy increment when flipping the spin of a configuration σ at a site x can be expressed in the form

$$\Delta_x H = H(\sigma^{(x)}) - H(\sigma) = [\tilde{J} M_J^{(x)}(\sigma) + K M_K^{(x)}(\sigma) + h]\, \sigma(x) \qquad (3.1)$$

where

$$M_J^{(x)}(\sigma) = \sum_{y: |x-y|=1} \sigma(y), \qquad M_K^{(x)}(\sigma) = \sum_{y: |y-x|=\sqrt{2}} \sigma(y)$$

We are interested in the region of the phase diagram where K is small with respect to $\tilde{J}$ (next nearest neighbor interaction is a perturbation of nearest neighbor interaction). Recall that we are assuming that

$$0 < 7h < K \leqslant \frac{J}{10} \tag{3.2}$$

Then every spin flip leads to a change of energy by a minimal amount of at least h.

Lemma 3.1. Suppose that $0 < h < K$ and $0 < K \leqslant \frac{1}{2}(\tilde{J} - h)$. Then, for a single spin flip,

$$|\Delta_x H| \geqslant h \tag{3.3}$$

Proof. Observe that $M_J^{(x)}$, $M_K^{(x)} \in \{-4, -2, 0, 2, 4\}$. By inspection one verifies that the minimal value is attained for $M_J = M_K = 0$. ∎

We will consider five particular classes of spin flips at a site x:

(a) An h-erosion if $\sigma(x) = +1$ and $M_J^{(x)} = M_K^{(x)} = 0$.
(b) An h-recovery if $\sigma(x) = -1$ and $M_J^{(x)} = M_K^{(x)} = 0$.
(c) A K-protuberance if $\sigma(x) = -1$, $M_J^{(x)} = 0$, and $M_K^{(x)} = -2$.
(d) A K-erosion if $\sigma(x) = +1$, $M_J^{(x)} = 0$, and $M_K^{(x)} = 2$.
(e) A J-protuberance if $\sigma(x) = -1$, $M_J^{(x)} = -2$, and $M_K^{(x)} = 0$.

Notice that according to (3.1), the energy increments in these cases are, respectively,

$$\Delta_x H(\sigma) = +h \tag{3.4a}$$

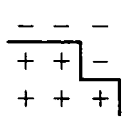
$$\Delta_x H(\sigma) = -h \tag{3.4b}$$

$$\Delta_x H(\sigma) = 2K - h \tag{3.4c}$$

$$\Delta_x H(\sigma) = 2K + h \tag{3.4d}$$

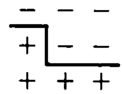
$$\Delta_x H(\sigma) = 2\tilde{J} - h \tag{3.4e}$$

For a site x adjacent to an isolated contour, an h-erosion is possible iff x is adjacent to a convex angle as in Scheme 3.1a (figures are always drawn modulo reflections and rotations).



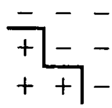
Scheme 3.1a

The site x is the central one. Similarly an h-recovery is possible iff x is adjacent to a concave angle as in Scheme 3.1b.



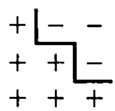
Scheme 3.1b

For a K-protuberance the situation is necessarily that in Scheme 3.1c.



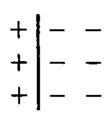
Scheme 3.1c

For a K-erosion it is that on Scheme 3.1d.



Scheme 3.1d

Finally, for a J-protuberance it is that on Scheme 3.1e.



Scheme 3.1e

Before discussing the "full dynamics" at nonzero temperature, we first consider a deterministic dynamics where only spin flips with decreasing energy are allowed. What we get is actually a certain generalization of *bootstrap percolation.* Consider finite sequences $S = \{x_1,..., x_n\}$ of lattice sites. A sequence S is called a *sweep* if it contains every site from Λ. The sequence S is called *standard* if it contains at least

$$(8\tilde{J} + 8K + 2h)\frac{M^2}{h} \tag{3.5}$$

consecutive sweeps. We use $\mathcal{S}$ to denote the set of all standard sequences.

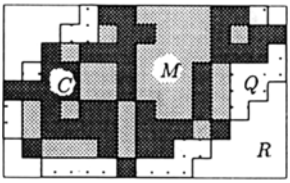
Every sequence S yields a map on configurations (denoted again S):

visit the sites in the order of S and flip the corresponding spin if it leads to a decrease of energy. Notice that the order in which the sites are visited is crucial. The particular number (3.5) of sweeps in a standard sequence S ensures that we always finally reach a local minimum of energy—a configuration that no longer allows a spin flip decreasing energy. Indeed, in every step when the energy changes, it decreases by at least h, the difference of absolute minimum and absolute maximum in energies is at most $(8\tilde{J}+8K+2h)\,M^2$, and finally, if no change during a sweep is made, the configuration is a local minimum.

As stated in Lemma 2.1, octagons are configurations forming local minima of the energy. Before we proceed with the investigation of the saddle points between octagons, we prove this lemma. However, first we introduce some concepts to be used in the proof and later.

We call a *droplet* any connected set C consisting of a union of unit squares centered at lattice sites. Clearly, for every droplet C there exists a configuration σ such that the droplet C is given by the set $\{x, \sigma(x) = +1\}$.

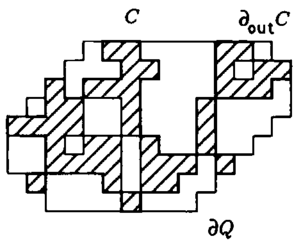
To every droplet we assign three envelopes, $M(C)$, $Q(C)$, and $R(C)$. The sets $R(C)$ and $Q(C)$, *rectangular* and *octagonal envelopes*, are the smallest rectangle and octagon, respectively, containing C. The set $M(C)$, the *monotone envelope* of C, is the smallest monotone droplet containing C, where a droplet is called *monotone* if its boundary has the same length as its rectangular envelope. (See Scheme 3.2.)



Scheme 3.2

Observe that if the rectangle $R = R(C)$ is not winding around the torus, the monotone envelope $M(C)$ is the complement of the union of all those open right angles of the form $\{(x, y) \in R; x > x_0, y > y_0\}$, $\{(x, y) \in R; x > x_0, y < y_0\}$, $\{(x, y) \in R;\ x < x_0, y > y_0\}$, and $\{(x, y) \in R;$ $x < x_0, y < y_0\}$ with (x_0, y_0) points of the dual lattice that do not intersect the set C. Notice that if $R(C)$ winds around the torus, then $M(C) = Q(C) = R(C)$.

Proof of Lemma 2.1. Consider an isolated connected component C of the set $\{x, \sigma(x) = +1\}$ and the octagon Q circumscribed to it. We will show that, supposing that the energy of σ increases under every spin flip, one necessarily has $C \equiv Q$. Consider first the *outer boundary* $\partial_{\text{out}} C$. To avoid ambiguities, it can be defined by taking the outer boundary of an ε-neighborhood of C in the limit $\varepsilon \to 0$. Taking into account that for $\varepsilon > 0$ one has a self-avoiding curve, we can consider (as will be useful later) a path winding around along all the boundary $\partial_{\text{out}} C$. Our aim now is to prove first that $\partial_{\text{out}} C \equiv \partial Q$ (see Scheme 3.3; the set C is shaded, the heavy line denotes $\partial_{\text{out}} C$).



Scheme 3.3.

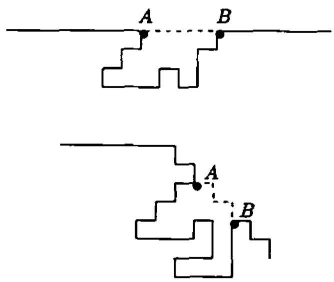
To this end we inspect a catalogue of locally stable configurations (with respect to spin flip at x). We say that a configuration σ is *stable at* x if the spin flip $\sigma(x) \to -\sigma(x)$ increases the energy. Consider thus a configuration σ and a site x with $\sigma(x) = -1$. Whether the configuration σ is stable at x depends only on its value at the nearest and next nearest neighbor sites of x. Namely, it is stable at x whenever either $M_J^{(x)} < 0$ or $M_J^{(x)} = 0$ and $M_K^{(x)} < 0$. As a result we get the following catalogue (up to rotations and reflections) of stable situations around x (dots stand for an arbitrary spin):

a)	b)	c)	d)
• • •	+ + −	− + −	− + +
− − −	− − −	− − −	− − +
• − •	− + −	− + −	− − −

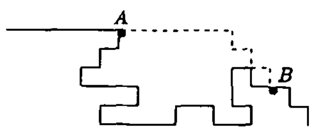
e)	f)	g)
+ + −	− + −	− + −
− − +	− − +	− − +
− − −	+ − −	− − −

Suppose now that $\partial_{\text{out}} C \neq \partial Q$. Then, considering a path along ∂Q oriented in the same sense as that along $\partial_{\text{out}} C$, there exist two points $A, B \in \partial_{\text{out}} C \cap \partial Q$

such that the paths along $\partial_{\rm out} C$ and ∂Q between these two points do not have a common unit segment and such that either (a) the points A, B belong to the same side of Q (see Scheme 3.4a) or (b) they belong to two neighboring sides of Q (see Scheme 3.4b).

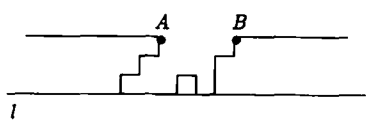


Scheme 3.4a



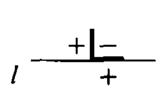
Scheme 3.4b

The easiest case to tackle is that of Scheme 3.4a with the side in question being, say, horizontal. Namely, consider the path γ between A and B and the lowermost horizontal line l touching γ (see Scheme 3.5).

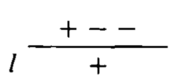


Scheme 3.5

At the point where γ for the first time (going from A to B) touches l, we have the configuration



For the spin $-$ to be stable one must have



and, l being the lowermost line touching γ, we must have

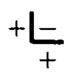
$$\begin{array}{l} \quad + \; - \; - \\ \hline l \qquad\; + \; + \end{array}$$

In view of (e) from our catalogue we necessarily have

$$\begin{array}{l} \quad - \; - \; - \\ \quad + \; - \; - \\ \hline l \quad - \; + \; + \end{array}$$

Otherwise the spin $-$ in the center would not be stable. The spin $+$ above the line l is, according to (a) from our catalogue, always unstable (it has three $-$ nearest neighbors) and we get a contradiction. In the remaining cases shown in Scheme 3.4 we get a contradiction by the same reasoning, once the lowermost horizontal line l touching γ does not pass through the point B.

The argument above can be also interpreted in the following way: whenever we encounter a concave corner



then, for the spin $-$ to be stable, we must have

$$\begin{array}{ccc} + & - & - \\ & + & \end{array}$$

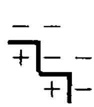
and hence

$$\begin{array}{ccc} + & - & - \\ & + & - \end{array}$$

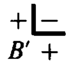
because supposing the configuration

$$\begin{array}{ccc} + & - & - \\ & + & + \end{array}$$

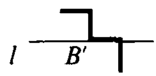
leads to a contradiction. Discussing in the same fashion also the value of the spin above the upper $+$, we conclude that the configuration necessarily is



and our concave corner is surrounded by two convex corners. We can use this observation in the following way. Consider the horizontal line l passing through B and the point B' where γ first hits l. Knowing already that γ does not pass below l, it proceeds from B' horizontally so that a concave corner is formed:

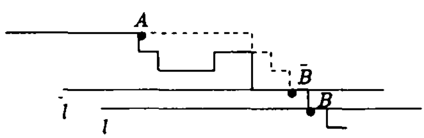


According to the above observation there must be two convex corners attached:



But since the line γ does not, before reaching B, pass below l, the point B' must coincide with B.

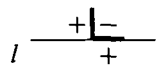
Consider now the point $\bar{B}$ on γ, two units backward from B, and repeat the above argument with the curve $\bar{\gamma}$ joining A with $\bar{B}$ and the horizontal line $\bar{l}$ passing through $\bar{B}$ (see Scheme 3.6).



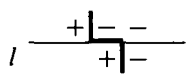
Scheme 3.6

If $\bar{l}$ also passes through A, the curve $\bar{\gamma}$ has to be just a horizontal segment joining A and $\bar{B}$. If not, we can repeat the argument and get a point $\bar{\bar{B}}$. Iterating this process, we eventually get a line passing through A. As a result, the curve γ actually follows the boundary ∂Q and we can conclude that $\partial_{\text{out}} \equiv \partial Q$.

To show, finally, that $\partial C = \partial_{\text{out}} C$ (there are no holes in C), consider the set $\partial C \setminus \partial_{\text{out}} C$ and the lowermost horizontal line touching it. Taking the left most touching point, we necessarily have a concave corner



and thus also



The newly attached bond lies below l and at the same time does not belong to $\partial_{out} C$; this would be possible only if the upper right spin were $+$, which is not the case. Hence we get a contradiction with the fact that the set $\partial C \backslash \partial_{out} C$ does not reach below l.

The fact that the octagon C has to be stable $(L_i, l_i \geqslant 2)$ is obvious. Finally, if the configuration σ contained two octagonal components of mutual distance one, there would always exist, as it is easy to convince oneself by inspection of possible cases, a minus spin between them, whose flipping would lead to a decrease of energy. ■

Among octagons with the same circumscribed rectangle, $Q = Q(D_1, D_2, l_1, ..., l_4)$, the standard ones, or at least those with oblique sides as close as possible to l^*, minimize the energy. This is stated, in a slightly more general form, in the following lemma.

Lemma 3.2. Let R be a rectangle with sides $D_1 \geqslant D_2$ and consider the set $\mathcal{M}(D_1, D_2)$ of all monotone droplets with connected interior[4] whose circumscribed rectangle is R. Let σ_0 be the configuration corresponding to:

(a) The octagon $Q(D_1, D_2, l^*)$ if $D_2 \geqslant 2l^* - 1$.

(b) The octagon $Q(D_1, D_2, l_1 = l_2 = l_3 = l_4 = \frac{1}{2}(D_2 + 1))$ if $D_2 < 2l^* - 1$ and it is odd.

(c) The octagon $Q(D_1, D_2, l_1 = l_2 = D_2/2, l_3 = l_4 = D_2/2 + 1)$ if $D_2 < 2l^* - 1$ and it is even.

Then

$$\min_{\sigma \in \mathcal{M}(D_1, D_2)} H(\sigma) = H(\sigma_0)$$

Proof. Let $\sigma \in \mathcal{M}(D_1, D_2)$ and consider its circumscribed octagon $Q(D_1, D_2, l_1, ..., l_4)$. The configuration $\bar{\sigma}$ represented by Q has clearly a lower energy than σ because it occupies a larger area then $C(\sigma)$ and, taking into account that in every site of the dual lattice at most two of its edges meet, its boundary necessarily has at least the same number of corners as $C(\sigma)$ (this is clearly true for every one of its oblique sides separately). The energy of $\bar{\sigma}$ is

$$H(\bar{\sigma}) = H(-\underline{1}) + 2J(D_1 + D_2) - hD_1 D_2 + \sum_{a=1}^{4} F(l_a)$$

[4] Notice that even though a monotone envelope is connected, its interior might split into disjoint components in a situation like that illustrated in Scheme 2.1. Cf. Lemma 4.3.

Notice that the function

$$F(l) \equiv -K(2l-1) + \tfrac{1}{2}hl(l-1) \tag{3.6}$$

is minimized for $l = l^*$. Indeed, for $l \in \mathbb{R}$, the parabola F has a minimum at $l = 2K/h + 1/2$ and this point is closer to l^* than to either $l^* - 1$ or $l^* + 1$. Hence, whenever $D_2 \geqslant 2l^* - 1$, the energy of $\bar{\sigma}$ is larger than or equal to that of σ_0 minimizing every term $F(l_a)$ separately.

If $D_2 < 2l^* - 1$, we first fix $\sum l_a$ and under this condition minimize $\sum F(l_a)$. The minimum is achieved for a maximally symmetric quadruple $l'_1,..., l'_4$, for which $\max_a l'_a - \min_a l'_a \leqslant 1$, such that $\sum l'_a = \sum l_a$. If $\sum l_a$ is not divisible by four, there is some freedom in the choice of $l'_1,..., l'_4$ and, in particular, the conditions

$$|l'_1 + l'_4 - (l'_2 + l'_3)| \leqslant 1 \qquad \text{and} \qquad |l'_1 + l'_2 - (l'_3 + l'_4)| \leqslant 1$$

can be satisfied. Given that $\sum l_a \leqslant 2(D_2 + 1)$, we have also

$$\max(l'_1 + l'_4, l'_2 + l'_3) \leqslant D_2 + 1 \text{ and } \max(l'_1 + l'_2, l'_3 + l'_4) \leqslant D_2 + 1 \leqslant D_1 + 1$$

and thus the octagon $Q(D_1, D_2, l'_1, l'_2, l'_3, l'_4)$ exists. We can further decrease its energy by increasing one by one $l'_1,..., l'_4$ maintaining maximal possible symmetry. The resulting octagon depends on the parity of D_2. ■

With every octagon Q we associate two *basins of attraction*—a narrow one

$$\mathscr{B}(Q) = \{\sigma: S\sigma = Q \text{ for every standard } S\} \tag{3.7}$$

and a wide one,

$$\hat{\mathscr{B}}(Q) = \{\sigma: \text{there exists a standard } S \text{ such that } S\sigma = Q\} \tag{3.8}$$

Clearly,

$$\hat{\mathscr{B}}(Q) \supset \mathscr{B}(Q)$$

and the sets $\mathscr{B}(Q)$ are disjoint for different Q's.

When discussing the growth of a droplet, we study the passage between close standard octagons with different rectangular envelopes. It is possible to pass between different octagons with the same rectangular envelope at the cost of overcoming energetic barriers between them. A crucial circumstance here is that the barriers are much lower than those one has to pass when changing the rectangular envelope—the corresponding time scales are much shorter and on the scale relevant for the passage to a "neighboring" standard octagon, these barriers can be overcome with a nonvanishing probability. If we choose to disregard the lower barriers, we can introduce "domains of attraction" associated with every standard octagon $Q = Q(D_1, D_2)$ specified by its rectangular envelope $R(D_1, D_2)$. Namely, we combine different octagons with the same rectangular envelope and introduce a broad *domain of attraction* $\hat{\mathscr{D}}(D_1, D_2)$ and a naturally restricted one $\mathscr{D}(D_1, D_2)$. We thus define

$$\hat{\mathscr{D}}(D_1, D_2) = \bigcup_{Q:(D_1(Q), D_2(Q)) = (D_1, D_2)} \hat{\mathscr{B}}(Q) \tag{3.9}$$

for every $(D_1, D_2) \in \mathbb{Z}_+^2$. For (D_1, D_2) such that $\min(D_1, D_2) \geq 3l^* - 2$ we define

$$\mathscr{D}(D_1, D_2) = \{\sigma\text{: for every standard } S, S\sigma = Q \text{ with } Q \text{ such that } (D_1(Q), D_2(Q)) = (D_1, D_2) \text{ and } \min(L_1(Q), L_2(Q)) \geq l^*\} \tag{3.10}$$

Notice that the condition $\min(D_1, D_2) \geq 3l^* - 2$ assures that the standard octagon $Q(D_1, D_2) \in \mathscr{D}(D_1, D_2)$. Let us introduce also the *boundaries*

$$\partial\mathscr{D} = \{\sigma \notin \mathscr{D} \text{ such that there exists } x \text{ such that } \sigma^{(x)} \in \mathscr{D}\} \tag{3.11}$$

and

$$\partial\hat{\mathscr{D}} = \{\sigma \in \hat{\mathscr{D}} \text{ such that there exists } x \text{ such that } \sigma^{(x)} \notin \hat{\mathscr{D}}\} \tag{3.12}$$

Notice that, even though we take $\partial\mathscr{D}$ outside $\mathscr{D}$, clearly $\partial\mathscr{D} \subset \hat{\mathscr{D}}$ and thus all configurations in $\partial\mathscr{D}$ and $\partial\hat{\mathscr{D}}$ can "fall down" to a minimum "inside" $R(D_1, D_2)$. To see that $\partial\mathscr{D} \subset \hat{\mathscr{D}}$, we observe that if $\xi \in \mathscr{D}$ and $\sigma = \xi^{(x)} \in \partial\mathscr{D}$, then $H(\xi) < H(\sigma)$. Indeed, if σ were stable at x, the flip $\xi \to \sigma$ would decrease energy and the sequence $\{x, S\}$ with S such that $S\sigma$ leads to Q with $(D_1(Q), D_2(Q)) \neq (D_1, D_2)$ or with $\min(L_1(Q), L_2(Q)) < l^*$ [such a sequence S exists since $\sigma \notin \mathscr{D}(D_1, D_2)$] would map the configuration ξ to this Q, in contradiction with the assumption $\xi \in \mathscr{D}$.

Whenever $\min(D_1, D_2) \geq 2l^* - 1$, we introduce the energy $\hat{E}(D_1, D_2)$ by

$$\hat{E}(D_1, D_2) = \begin{cases} h(\min(D_1, D_2) - 1) - 4K & \text{if } \min(D_1, D_2) < D^* \\ 2J - 4K - h & \text{if } \min(D_1, D_2) \geq D^* \end{cases} \tag{3.13}$$

and, for $\min(D_1, D_2) \geq 3l^* - 2$, also

$$E(D_1, D_2) = \hat{E}(D_1, D_2) + \max(0, 3l^* - \min(D_1, D_2))\,\eta h \tag{3.14}$$

where η is the constant defined by (2.18). For sufficiently large $\min(D_1, D_2)$ we have $E(D_1, D_2) = \hat{E}(D_1, D_2)$, and according to the following lemma, this energy actually yields the height, above the value $H(Q(D_1, D_2))$ of the energy of the standard octagon, of the lowest point on the boundary of the domains of attraction $\mathscr{D}(D_1, D_2)$ and $\hat{\mathscr{D}}(D_1, D_2)$ as well as the heights of the saddle points for escaping paths. If $\min(D_1, D_2) < 3l^*$, the situation is more complex. A path leaving $\mathscr{D}$ through a saddle point is actually heading to an octagon Q still belonging to $\hat{\mathscr{D}}$—the saddle point is not on the boundary of $\hat{\mathscr{D}}$.

Lemma 3.3. For every $(D_1, D_2) \in \mathbb{Z}_+^2$ such that $\min(D_1, D_2) \geqslant 2l^* - 1$ one has

$$\min_{\omega: \sigma_0 \to \mathscr{M}^c} \sup_{\sigma \in \omega} H(\sigma) < > \min_{\sigma \in \partial\hat{\mathscr{D}}(D_1, D_2)} H(\sigma) = H(Q(D_1, D_2)) + \hat{E}(D_1, D_2) \tag{3.15}$$

where the minimum is over all paths starting at any octagon $Q = Q(D_1, D_2, l_1, l_2, l_3, l_4)$ and leaving the set $\mathscr{M}$ of all monotonic configurations.

If, moreover, $\min(D_1, D_2) \geqslant 3l^* - 2$, then also

$$\min_{\omega: \sigma_0 \to \mathscr{D}^c} \sup_{\sigma \in \omega} H(\sigma) = \min_{\sigma \in \partial\mathscr{D}(D_1, D_2)} H(\sigma) = H(Q(D_1, D_2)) + E(D_1, D_2) \tag{3.16}$$

where the minimum is over all paths starting at the standard octagon $Q(D_1, D_2)$ and leaving the set $\mathscr{D}(D_1, D_2)$.

Proof. Let $\sigma \in \partial\hat{\mathscr{D}}$, $\xi = \sigma^{(x)} \notin \hat{\mathscr{D}}$. Since $\sigma \in \hat{\mathscr{D}}$, there exists a standard S such that $S\sigma = Q$ and $(D_1(Q), D_2(Q)) = (D_1, D_2)$. Consider the sequence of configurations obtained by applying S on σ. Taking it in the opposite order, we get a path $\omega: Q \to \sigma$ such that the energy increases in every step.

With $Q = Q(D_1, D_2, l_1, l_2, l_3, l_4)$, $D_1 \geqslant D_2$, and using the function F defined in (3.6), we shall prove that

$$H(\sigma) \geqslant H(Q) + \min[h(D_2 - 1) - 4K, 2J - 4K - h] - \sum_{\alpha=1}^{4} [F(l_\alpha) - F(l^*)] \tag{3.17}$$

On the other hand, particular cases of $\sigma \in \partial\hat{\mathscr{D}}$ yielding equality in (3.17) can be displayed. Namely, if $D_2 \leqslant D^*$, we take the standard octagon $Q(D_1, D_2)$ and cut, except for one spin, all of the row of $(D_2 - 1)$ spins; the resulting droplet corresponds to $\sigma \in \partial\hat{\mathscr{D}}$ since cutting the last spin we get $\xi \notin \hat{\mathscr{D}}$. If $D_2 > D^*$, we get $\sigma \in \partial\hat{\mathscr{D}}$ just by attaching one plus spin to the coordinate side.

Taking into account that

$$H(Q) = H(-\underline{1}) + 2J(D_1 + D_2) - hD_1 D_2 + \sum_{a=1}^{4} F(l_a) \tag{3.18}$$

we get from (3.17) the sought minimum for $\partial\hat{\mathscr{D}}(D_1, D_2)$,

$$\min_{\sigma \in \partial\hat{\mathscr{D}}(D_1, D_2)} H(\sigma) = H(Q(D_1, D_2)) + \hat{E}(D_1, D_2)$$

To prove (3.17), notice first that, without loss of generality, we can suppose that the path ω consists of at most $D_2 - 1 - \max(l_1 + l_2 - 2, 2l^* - 2)$ steps. [For concreteness we suppose that $\max(l_1 + l_2, l_2 + l_3, l_3 + l_4, l_4 + l_1) = l_1 + l_2$.] Indeed, in every step the energy increases by at least h and

$$h(D_2 - (2l^* - 1)) > h(D_2 - 1) - 4K$$

since $h(l^* - 1) < 2K$. Hence, the number of steps can be taken to be at most $D_2 - 1 - 2(l^* - 1)$. If $l_1 + l_2 > 2l^*$, we have an even stronger restriction on the number of steps. Indeed, it suffices to prove a lower bound on the increase of energy,

$$h[D_2 - 1 - (l_1 - 1) - (l_2 - 1)] \geqslant h(D_2 - 1) - 4K - \sum_{a=1}^{4} [F(l_a) - F(l^*)] \tag{3.19}$$

This is true once we verify that

$$F(l) - F(l^*) \geqslant h(l - 1) - 2K \tag{3.20}$$

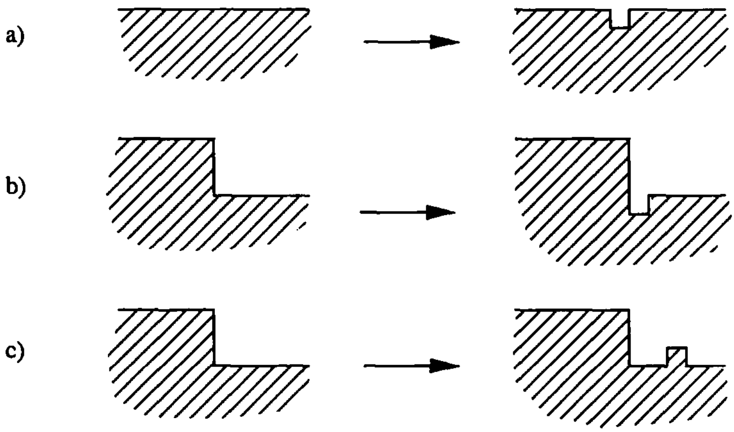
for any l. To see this, we observe that the line (as a function of l) on the right-hand side above touches the parabola on the left-hand side in the point $l = l^* + 1$ and is below it for $l = l^*$ and $l = l^* + 2$:

$$l = l^*: \quad F(l) - F(l^*) = 0 > h(l^* - 1) - 2K$$

$$l = l^* + 1: \quad F(l) - F(l^*) = -2K + \tfrac{1}{2} h l^* 2 = -2K + hl^*$$

$$l = l^* + 2: \quad F(l) - F(l^*) = -4K + h(2l^* + 1) > -2K + h(l^* + 1)$$

[Notice that equality in (3.19) is attained only for $l_1 = l_2 = l^* + 1$.]



Scheme 3.7

Next, we can suppose that all droplets $\zeta \in \omega$ (including the last one, σ) are monotone with connected interior. Indeed, starting from the octagon Q and supposing that after $n \leqslant D_2 + 1 - l_1 - l_2$ steps we still have such a monotone droplet, minimal energy flips leading to a nonmonotone droplet are those shown in Scheme 3.7. (The given number of steps is clearly not sufficient to allow any configuration like

with subsequent splitting into disjoint components, etc.) The increase of energy in the cases (a), (b), and (c) is $2J - 4K + h$, $2J - 2K + h$, and $2J - 4K - h$, respectively. In all three cases this value is at least as large as $\min[h(D_2 - 1) - 4K, 2J - 4K - h]$ and already this single step would suffice for our claim. This remark proves, in particular, the first inequality in (3.15).

Observe now that the spin flip $\sigma \to \sigma^{(x)} = \xi$ necessarily decreases energy (otherwise one would have $\xi \in \hat{\mathscr{D}}$). Taking into account that σ is monotone with connected interior, the droplet ξ is also monotone with interior consisting of at most two components. Suppose first that ξ has connected interior. For such ξ, there clearly exists a standard sequence leading to its octagonal envelope $Q(\xi)$ and, since $\xi \notin \hat{\mathscr{D}}$, one has $(D_1(\xi), D_2(\xi)) \neq (D_1, D_2)$. As a consequence

either $(D_1(\sigma), D_2(\sigma)) \neq (D_1, D_2)$ [actually $D_1(\sigma) > D_1$ or $D_2(\sigma) > D_2$] or $(D_1(\sigma), D_2(\sigma)) = (D_1, D_2)$ and $\bar{d}_a = 1$ for some $a = 1,..., 4$

[we use here $\bar{d}_a(\sigma)$ to denote the lengths of segments along which the set $C(\sigma)$ intersects the sides of the circumscribed rectangle $R(\sigma)$].

Consider the first configuration ζ in ω with this property and its predecessor $\bar{\zeta}$. If $D_1(\zeta) > D_1$ or $D_2(\zeta) > D_2$, then

$$\begin{aligned} H(\sigma) - H(Q) &\geqslant H(\zeta) - H(\bar{\zeta}) \geqslant 2J - 4K - h \\ &\geqslant \min[h(D_2 - 1) - 4K, 2J - 4K - h] \end{aligned} \tag{3.21}$$

If $(D_1(\zeta), D_2(\zeta)) = (D_1, D_2)$ and at the same time $\bar{d}_a = 1$, one had to cut at least $L_a - \bar{d}_a = L_a - 1$ spins touching the side of $R(Q)$ to reach this configuration, and thus

$$H(\zeta) - H(Q) \geqslant h(L_a - 1) \geqslant h(D_2 - 1 - (l_1 - 1) - (l_2 - 1)) \tag{3.22}$$

Hence, using again (3.19), we get (3.17).

If the interior of ξ consists of two components, then the path ω reaching the configuration σ should consist of at least $L_a - 1$ steps. Indeed, consider the horizontal (or vertical) line passing through the point in which the closures of these two components intersect. In the configuration Q there are at least $2L_a$ plus spins at sites of distance 1/2 from this line, while in the configuration ξ at least L_a of them are missing ("two quadrants filled with minuses touch at the considered intersection point"). As a result, the inequality (3.22) holds for σ and we get (3.17).

To get the bound for $\partial\mathscr{D}(D_1, D_2)$, consider first $\sigma \in \partial\mathscr{D}(D_1, D_2) \backslash \partial\hat{\mathscr{D}}(D_1, D_2)$. Notice that then there exists a standard sequence S mapping σ onto an octagon Q, $S\sigma = Q$, such that $\min(L_1(Q), L_2(Q)) < l^*$. Indeed, by the same argument as above we can show that all ζ on the (reversed) path from Q to σ are monotone, $(D_1(\sigma), D_2(\sigma)) = (D_1, D_2)$, and $\bar{d}_a(\sigma) \geqslant 2$, $a = 1,..., 4$ [otherwise we would have $\sigma \in \partial\hat{\mathscr{D}}(D_1, D_2)$]. Hence, there does not exist any standard sequence S mapping σ onto an octagon Q such that $(D_1(Q), D_2(Q)) \neq (D_1, D_2)$.

Moreover, for at least one ζ on our path one has $Q(\zeta) \supset Q$ and $Q(\zeta) \neq Q$. There exists a standard sequence mapping ζ into Q and thus $Q(\zeta) \supset Q$. If one hand $Q(\zeta) = Q$ for every ζ including σ, then one could use the fact that, since $\sigma \in \partial\mathscr{D}$, there exists a standard sequence $\bar{S}$ mapping σ into $\bar{Q}$ such that $(D_1(\bar{Q}), D_2(\bar{Q})) = (D_1, D_2)$ and $\min(L_1(\bar{Q}), L_2(\bar{Q})) \geqslant l^*$ to get a contradiction. Indeed, for such $\bar{Q}$ one cannot have $\bar{Q} \subset Q(\sigma) = Q$ and at the same time $(D_1(\bar{Q}), D_2(\bar{Q})) = (D_1(Q), D_2(Q))$. Thus, on the uphill path from Q to σ a K-protuberance appears at least once and thus

$$H(\sigma) \geqslant H(Q) + 2K - h \tag{3.23}$$

We can label the sides of the octagon Q in such a way that $L_2 = D_2 - l_1 - l_2 + 2 \leqslant l^* - 1$, which is equivalent to

$$l_1 + l_2 - 2 \geqslant D_2 - 1 - (l^* - 2) \tag{3.24}$$

The difference of the energy of the octagon Q and the standard octagon $Q(D_1, D_2)$ is at least

$$H(Q) - H(Q(D_1, D_2)) \geqslant \sum_{a=1}^{4} (F(l_a) - F(l^*)) \tag{3.25}$$

If $\min(D_1, D_2) \geqslant 3l^* - 1$, we use the bound (3.20) to evaluate, using (3.24), the right-hand side

$$\begin{aligned}\sum_{a=1}^{4} (F(l_a) - F(l^*)) &\geqslant F(l_1) - F(l^*) + F(l_2) - F(l^*) \\ &\geqslant (l_1 + l_2 - 2)\, h - 4K \\ &\geqslant (D_2 - 1)\, h - 4K - (l^* - 2)\, h \end{aligned} \tag{3.26}$$

With the help of the equality (2.18), $\eta h = 2K - h - (l^* - 2)\, h$, we get from (3.23) and the above inequality the sought lower bound with 1 in place of $3l^* - \min(D_1, D_2)$ in the definition (3.14).

If $\min(D_1, D_2) = D_2 = 3l^* - 2$, the bound (3.24) asserts that

$$l_1 + l_2 - 2 \geqslant 2(l^* - 1) + 1 \tag{3.27}$$

Supposing, say, $\max(l_1, l_2) = l_1$, we can infer that

$$l_1 \geqslant l^* + 1 \tag{3.28}$$

and thus

$$\sum_{a=1}^{4} (F(l_a) - F(l^*)) \geqslant F(l_1) - F(l^*) \geqslant hl^* - 2K \tag{3.29}$$

Using now the bound (3.23) and then the bound (2.17) and the definition (2.18), we get

$$\begin{aligned} H(\sigma) &\geqslant H(Q(D_1, D_2)) + hl^* - 2K + 2K - h \\ &= H(Q(D_1, D_2)) + h(l^* - 1) \\ &= H(Q(D_1, D_2)) + h(3l^* - 3) - 4K + 2\eta h \end{aligned} \tag{3.30}$$

Thus, we have finished the proof for $\sigma \in \partial\mathscr{D}(D_1, D_2) \backslash \partial\hat{\mathscr{D}}(D_1, D_2)$.

Let us now turn to the case $\sigma \in \partial\mathscr{D}(D_1, D_2) \cap \partial\hat{\mathscr{D}}(D_1, D_2)$. If $\min(D_1, D_2) \geqslant 3l^*$, the needed bound is the already proved inequality (3.15).

In the case $(D_2 =)\ \min(D_1, D_2) = 3l^* - 1$, one can consider a path leading to Q with $L_2(Q) \geqslant l^*$. In the case leading to (3.21) the needed bound is amply satisfied since

$$D_2 h - 4K + 2h < 2J - 4K - h \tag{3.31}$$

In the case leading to (3.22) we get

$$H(\sigma) - H(Q) \geqslant h(L_2 - 1) \geqslant h(l^* - 1) \geqslant h(3l^* - 2) - 4K + \eta h \tag{3.32}$$

using (2.17).

If $(D_2 =)\ \min(D_1, D_2) = 3l^* - 2$, we get again either the sufficient bound (3.21) or

$$H(\sigma) - H(Q) \geqslant h(L_2 - 1) \geqslant h(l^* - 1) = h(3l^* - 3) - 4K + 2\eta h \tag{3.33}$$

Finally, to prove that

$$\min_{\omega: \sigma_0 \to \mathscr{D}^c} \sup_{\sigma \in \omega} H(\sigma) = H(Q(D_1, D_2)) + E(D_1, D_2) \tag{3.34}$$

we have to find a path ω from the standard octagon $Q(D_1, D_2)$ to a saddle configuration σ with

$$H(\sigma) = H(Q(D_1, D_2)) + E(D_1, D_2)$$

such that for every $\zeta \in \omega$ one has

$$H(\zeta) \leqslant H(Q(D_1, D_2)) + E(D_1, D_2)$$

If $\min(D_1, D_2) \geqslant 3l^*$, the saddle point σ is actually on the boundary $\partial\hat{\mathscr{D}}(D_1, D_2)$.

A path satisfying the above condition can be taken, for example, by first cutting one layer along two oblique sides of the standard octagon and then along the coordinate side between them. The highest point on this path, before the final steady growth when cutting the coordinate side, is

$$\begin{aligned} &H(Q(D_1, D_2)) + 2(l^* - 1)\, h - (2K - h) \\ &\quad < H(Q(D_1, D_2)) + (\min(D_1, D_2) - 1)\, h - 4K \end{aligned}$$

once $\min(D_1, D_2) \geqslant 3l^*$.

If $\min(D_1, D_2) = 3l^* - 1$, corresponding to $\min(L_1, L_2) = l^* + 1$, one reaches the saddle point of $\partial\mathscr{D}(D_1, D_2)$ already when finishing the cut of two oblique sides attached to the shorter coordinate side (yielding the coordinate side of the length $l^* - 1$) with the energy

$$H(Q(D_1, D_2)) + 2(l^* - 1)\, h - (2K - h) = H(Q(D_1, D_2)) + \hat{E}(D_1, D_2) + \eta h$$

Finally, if $\min(D_1, D_2) = 3l^* - 2$, corresponding to $\min(L_1, L_2) = l^*$, one reaches, already when cutting only one oblique side, the saddle point with the energy

$$H(Q(D_1, D_2)) + (l^* - 1)\, h = H(Q(D_1, D_2)) + \hat{E}(D_1, D_2) + 2\eta h \quad \blacksquare$$

In the following Lemma we evaluate, using the reversibility of the process, the hitting time τ_η to a configuration η, starting from a configuration σ, in terms of the difference of energies of the concerned configurations.

Lemma 3.4. For every $\varepsilon > 0$ and all $\sigma \neq \eta \in \Gamma$, $H(\sigma) < H(\eta)$, one has

$$P_\sigma(\tau_\eta \leqslant \exp\{\beta(H(\eta) - H(\sigma) - \varepsilon)\}) \xrightarrow[\beta \to \infty]{} 0$$

Proof. Given $T \in \mathbb{N}$, one has

$$\begin{aligned}
&P_\sigma(\tau_\eta < T)\\
&\quad = \sum_{0 < s < T} \sum_{\bar{\sigma}_1, \ldots, \bar{\sigma}_{s-1} \in \Gamma \setminus \eta} P_\sigma(\sigma_0 = \sigma, \sigma_1 = \bar{\sigma}_1, \ldots, \sigma_{s-1} = \bar{\sigma}_{s-1}, \sigma_s = \eta)\\
&\quad = \exp\{-\beta(H(\eta) - H(\sigma))\}\\
&\qquad \times \sum_{0 < s < T} \sum_{\bar{\sigma}_1, \ldots, \bar{\sigma}_{s-1} \in \Gamma \setminus \eta} P(\sigma_0 = \eta, \sigma_1 = \bar{\sigma}_{s-1}, \ldots, \sigma_{s-1} = \bar{\sigma}_1, \sigma_s = \sigma)\\
&\quad \leqslant T \exp[-\beta(H(\eta) - H(\sigma))] \quad \blacksquare
\end{aligned}$$

Our next task is to characterize octagons with respect to their tendency to grow or shrink. This will be done in Propositions 1–4. Before presenting them, however, we introduce some tools that will be repeatedly used.

First, we define a set of Markov chains obtained by observing our stochastic trajectories only when they pass through a particular set of configurations.

Let Q be a local minimum (octagon) of the energy and B any connected set of configurations containing Q. Here connected means that, for all σ, $\eta \in B$, there exists a sequence $\sigma_1 \cdots \sigma_k \in B$ such that $\sigma_1 = \sigma$, $\sigma_k = \eta$, and (σ_i, σ_{i+1}) are "nearest neighbor configurations" for $i = 1,..., k-1$, with σ and τ considered to be nearest neighbors iff there exists $x \in \Lambda$ such that $\tau = \sigma^{(x)}$.

The *exterior boundary* ∂B of B [cf. (3.11)] is the set

$$\partial B = \{\eta = \sigma^{(x)}; \eta \notin B, \sigma \in B, x \in \Lambda\} \tag{3.35}$$

For a particular Q and B we consider the following Markov chain, $\{\xi_n\}$, with the space of states

$$X = Q \cup \partial B$$

Introducing the sequence of times

$$v_0 < u_0 \leqslant v_1 < u_1 \leqslant v_2 < \cdots$$

with $v_0 = 0$, u_i, $v_i \in \mathbb{N}$,

$$\begin{aligned} u_n &= \inf\{t > v_n : \sigma_t \neq \sigma_{t-1}\} \\ v_n &= \inf\{t \geqslant u_n : \sigma_t \in \partial B \cup Q\} \end{aligned} \tag{3.36}$$

we set

$$\xi_n = \sigma_{v_n}, \qquad \sigma_0 \in Q \cup \partial B \tag{3.37}$$

and

$$\begin{aligned} \theta &= \inf\{n: \xi_n = Q\} \\ v &= \inf\{n \geqslant \theta: \xi_n \in \partial B\} \end{aligned} \tag{3.38}$$

For every $s \in \mathbb{N}$ one has

$$P_Q(\tau_{\partial B} > s) \geqslant P_Q(v > s) = P(Q \to Q)^s = [1 - P(Q \to \partial B)]^s$$

where

$$\begin{aligned} P(Q \to Q) &= P(\xi_1 = Q \mid \xi_0 = Q) \\ P(Q \to \partial B) &= \sum_{\eta \in \partial B} P(\xi_1 = \eta \mid \xi_0 = Q) = P(\xi_1 \in \partial B \mid \xi_0 = Q) \end{aligned} \tag{3.39}$$

We have

$$
\begin{aligned}
&P(\xi_1 \in \partial B \mid \xi_0 = Q) \\
&\quad = \sum_{s_0=0}^{\infty} \sum_{s=1}^{\infty} \sum_{\bar\sigma_1,\dots,\bar\sigma_{s-1} \in B\setminus\{Q\}} \sum_{\bar\sigma_s \in \partial B} P(\sigma_0 = Q,\dots, \sigma_{s_0} = Q, \\
&\qquad \sigma_{s_0+1} = \bar\sigma_1,\dots, \sigma_{s_0+s} = \bar\sigma_s) \\
&\quad = \sum_{s_0=1}^{\infty} \sum_{s=1}^{\infty} \sum_{\bar\sigma_1,\dots,\bar\sigma_{s-1} \in B\setminus\{Q\}} \sum_{\bar\sigma_s \in \partial B} \exp\{-\beta[H(\bar\sigma_s) - H(Q)]\} \\
&\qquad \times P(\sigma_0 = \bar\sigma_s, \sigma_1 = \bar\sigma_{s-1},\dots, \sigma_{s-1} = \bar\sigma_1, \sigma_s = Q,\dots, \sigma_{s+s_0} = Q) \\
&\quad \leqslant \exp\{-\beta \min_{\sigma\in\partial B} [H(\sigma) - H(Q)]\} \\
&\qquad \times \sum_{\bar\sigma \in \partial B} \sum_{s_0=1}^{\infty} \sum_{s=1}^{\infty} P_{\bar\sigma}(\sigma_t \in B\setminus Q, \forall t < s, \sigma_t = Q, \forall t \in [s, s+s_0]) \\
&\quad \leqslant |\partial B| \exp\{-\beta \min_{\sigma\in\partial B} [H(\sigma) - H(Q)]\}\, P_\sigma(\tau_Q \geqslant 1)
\end{aligned}
\tag{3.40}
$$

In conclusion, for all $\varepsilon > 0$ one has

$$
P(Q \to \partial B) < \exp\{-\beta[\min_{\sigma\in\partial B}(H(\sigma) - H(Q)) - \varepsilon]\}
\tag{3.41}
$$

whenever β is large enough.

For a set of octagons, say Q_j, $j = 1,\dots, N$, we will consider suitable connected (mutually disjoint) sets B_j containing Q_j and, as above, we introduce simultaneously the corresponding Markov chains ξ_n^j [see (3.37)] with the hitting times v_j, θ_j defined as in (3.38). To avoid ambiguities (the stochastic trajectory can come back to an octagon Q_j with $j > i$), the process ξ^j is defined considering the portion of the trajectory between the moment it first enters ∂B_j until, after visiting Q_j, it first enters ∂B_{j+1} (visiting, possibly, other B_i with $i < j$ in the meantime).

Often we will combine two subsequent events with the help of the following composition operator. In general, consider two events $\mathscr{E}_{t_1}^1$, $\mathscr{E}_{t_2}^2$ taking place in the intervals of time $[0, t_1]$, $[0, t_2]$, respectively. This means that $\mathscr{E}_{t_1}^1$, $\mathscr{E}_{t_2}^2$ are subsets of Ω measurable with respect to the σ-algebra generated by $\{\sigma_t\}_{t\in[0,t_1]}$, $\{\sigma_t\}_{t\in[0,t_2]}$, respectively. We define the *composition* $(\mathscr{E}_{t_1}^1; \mathscr{E}_{t_2}^2)$ of $\mathscr{E}_{t_1}^1$, $\mathscr{E}_{t_2}^2$ by taking

$$
(\mathscr{E}_{t_1}^1; \mathscr{E}_{t_2}^2) = \mathscr{E}_{t_1}^1 \cap T_{t_1} \mathscr{E}_{t_2}^1
\tag{3.42}
$$

with T_t denoting the translation operator shifting the events in a natural way by a time t.

More generally, given the events $\mathscr{E}^1_{t_1},\ldots,\mathscr{E}^n_{t_n}$ taking place in the intervals $[0, t_1],\ldots,[0, t_n]$, we set

$$(\mathscr{E}^1_{t_1};\mathscr{E}^2_{t_2};\ldots;\mathscr{E}^n_{t_n})=\mathscr{E}^1_{t_1}\cap T_{t_1}\mathscr{E}^2_{t_2}\cap\cdots\cap T_{t_1+t_2+\cdots+t_{n-1}}\mathscr{E}^n_{t_n} \tag{3.43}$$

Given $\mathscr{E}_t$ for every $t\in\mathbb{N}$ and a value $\bar{t}\in\mathbb{N}$, we set

$$\bar{\mathscr{E}}_{\bar{t}}=\bigcup_{t=1}^{\bar{t}}\mathscr{E}_t \tag{3.44}$$

We call $\bar{\mathscr{E}}_{\bar{t}}$ the *extension* of $\mathscr{E}_t$ up to $\bar{t}$.

Finally, we use $\bar{\mathscr{E}}$ to denote the extension $\bar{\mathscr{E}}_\infty$. Namely, we simply write

$$\bar{\mathscr{E}}=\bigcup_{t=1}^{\infty}\mathscr{E}_t \tag{3.45}$$

Given $\mathscr{E}^1_{t_1},\ldots,\mathscr{E}^n_{t_n}$, which we suppose to be defined for any n-tuple $t_1,\ldots,t_n\in\mathbb{N}$, we define the *composition of the extensions* $\bar{\mathscr{E}}^i_{\bar{t}_i}$, $i=1,\ldots,n$, as

$$\begin{aligned}(\bar{\mathscr{E}}^1_{\bar{t}_1};\bar{\mathscr{E}}^2_{\bar{t}_2};\ldots;\bar{\mathscr{E}}^n_{\bar{t}_n})&=\bigcup_{t_1=1}^{\bar{t}_1}\cdots\bigcup_{t_n=1}^{\bar{t}_n}(\mathscr{E}^1_{t_1};\ldots;\mathscr{E}^n_{t_n})\\&=\bigcup_{t_1=1}^{\bar{t}_1}\cdots\bigcup_{t_n=1}^{\bar{t}_n}[\mathscr{E}^1_{t_1}\cap T_{t_1}\mathscr{E}^2_{t_2}\cap\cdots\cap T_{t_1+t_2+\cdots+t_{n-1}}\mathscr{E}^n_{t_n}]\end{aligned} \tag{3.46}$$

As mentioned above, we will be interested in a process passing through a series of local energetic minima characterized by octagons Q_j, $j=1,\ldots,N$. In a typical situation, the boundaries of subsequent connected sets $B_1,\ldots,B_N$ intersect. Namely, we suppose that $\partial B_i\cap\partial B_{i+1}\neq 0$ for $i=1,\ldots,N-1$ and that a sequence of configurations $S_1, S_2,\ldots,S_{N+1}$ is given such that $S_{i+1}\in\partial B_i\cap\partial B_{i+1}$ for $i=1,\ldots,N-1$, $S_1\in\partial B_1$, and $S_{N+1}\in\partial B_N$. One can think about the configuration S_{i+1} as about a saddle point for the passage *up* from Q_i to S_{i+1} and then *down* from S_{i+1} to Q_{i+1}. Supposing further that a sequence $t^1_u, t^1_d,\ldots,t^{N-1}_d, t^N_u\in\mathbb{N}$ is given, with u and d standing for *up* and *down*, we define, for $j=1,\ldots,N$, the event

$$\begin{aligned}\mathscr{F}^{j,u}_{t^j_u}&=\{\omega\in\Omega\colon\sigma_0=Q_j,\ \tau_{\partial B_j}=t^j_u,\ \sigma_{\tau_{\partial B_j}}=S_{j+1}\}\\&=\{\xi^j_0=Q_j,\ \xi^j_{v_j}=S_{j+1},\ v_{v_j}=t^j_u\}\end{aligned} \tag{3.47}$$

("ascent up to the saddle point")[5] and, for $j=1,\ldots,N-1$, the event

$$\mathscr{F}^{j,d}_{t^j_d}=\{\omega\in\Omega\colon\sigma_0=S_{j+1},\ \tau_{\partial Q_{j+1}}=t^j_d\}=\{\xi^{j+1}_0=S_{j+1},\ v_{\theta_{j+1}}=t^j_d\} \tag{3.48}$$

[5] Notice that, since $\xi^j_{v_j}=S_{j+1}\in\partial B_j\cap\partial B_{j+1}$, the definition of the event $\mathscr{F}^{j,u}_{t^j_u}$ contains a condition that the stochastic trajectory is *not* revisiting any set B_i with $i<j$ before reaching ∂B_{j+1}.

("descent from the saddle point"). Further, we can consider the composition

$$(\mathscr{F}^{1,u}_{t^1_u}; \mathscr{F}^{1,d}_{t^1_d}; \mathscr{F}^{2,u}_{t^2_u}; \mathscr{F}^{2,d}_{t^2_d};...; \mathscr{F}^{N-1,d}_{t^{N-1}_d}; \mathscr{F}^{N,u}_{t^N_u}) \tag{3.49}$$

defined by Eq. (3.46). Given $\bar{t}^i_u$, $\bar{t}^i_d$, $i=1,..., N-1$, $\bar{t}^N_u \in \mathbb{N}$, we consider also the extensions up to $\bar{t}^i_u$, $\bar{t}^i_d$:

$$(\bar{\mathscr{F}}^{1,u}_{\bar{t}^1_u};...; \bar{\mathscr{F}}^{N-1,d}_{\bar{t}^{N-1}_d}; \bar{\mathscr{F}}^{N,u}_{\bar{t}^N_u}) \equiv \bigcup_{t^1_u=1}^{\bar{t}^1_u} \cdots \bigcup_{t^{N-1}_d=1}^{\bar{t}^{N-1}_d} \bigcup_{t^N_u=1}^{\bar{t}^N_u} (\mathscr{F}^{1,u}_{t^1_u};...; \mathscr{F}^{N-1,d}_{t^{N-1}_d}; \mathscr{F}^{N,u}_{t^N_u}) \tag{3.50}$$

Observing that the events $(\mathscr{F}^{1,u}_{t^1_u};...; \mathscr{F}^{N,u}_{t^N_u})$ in the right-hand side of Eq. (3.50) are manifestly disjoint, we have

$$\begin{aligned} &P(\bar{\mathscr{F}}^{1,u}_{\bar{t}^1_u};...; \bar{\mathscr{F}}^{N-1,d}_{\bar{t}^{N-1}_d}; \bar{\mathscr{F}}^{N,u}_{\bar{t}^N_u}) \\ &\quad = \sum_{t^1_u=1}^{\bar{t}^1_u} \cdots \sum_{t^{N-1}_d=1}^{\bar{t}^{N-1}_d} \sum_{t^N_u=1}^{\bar{t}^N_u} P(\mathscr{F}^{1,u}_{t^1_u};...; \bar{\mathscr{F}}^{N-1,d}_{t^{N-1}_d}; \mathscr{F}^{N,u}_{t^N_u}) \\ &\quad \equiv \left[\sum_{t^1_u=1}^{\bar{t}^1_u} P(\mathscr{F}^{1,u}_{t^1_u})\right] \cdots \left[\sum_{t^{N-1}_d=1}^{\bar{t}^{N-1}_d} P(\mathscr{F}^{N-1,d}_{t^{N-1}_d})\right]\left[\sum_{t^N_u=1}^{\bar{t}^N_u} P(\mathscr{F}^{N,u}_{t^N_u})\right] \\ &\quad \equiv P(\bar{\mathscr{F}}^{1,u}_{\bar{t}^1_u}) \cdots P(\bar{\mathscr{F}}^{N-1,d}_{\bar{t}^{N-1}_d})\, P(\bar{\mathscr{F}}^{N,u}_{\bar{t}^{N-1}_u}) \end{aligned} \tag{3.51}$$

We want now to restrict ourselves to a sequence of ascent events that can be defined only in terms of the above Markov chains, more precisely, in terms of the events whose characteristic functions are measurable with regard to the σ-algebra generated by the random variable ξ^i_n. This will be achieved by using

$$\bar{\mathscr{F}}^{j,u}_{t^j_u} = \bar{\mathscr{F}}^{j,u} \cap \{\sigma_0 = Q_j, \tau_{\partial B_j} \leqslant \bar{t}^j_u\} \tag{3.52}$$

where the event

$$\bar{\mathscr{F}}^{j,u} = \bigcup_{t^j_u=1}^{\infty} \mathscr{F}^{j,u}_{t^j_u} = \{\xi^j_0 = Q_j, \xi^j_{v_j} = S_{j+1}\}, \qquad j=1,..., N \tag{3.53}$$

has the desired measurability property. Introducing now

$$\mathscr{G} = \bigcap_{j=1}^{N-1} \{\tilde{\tau}_{\partial B_j} \leqslant \bar{t}^j_u\} \tag{3.54}$$

where $\tilde{\tau}_{\partial B_j}$ is defined by

$$\tilde{\tau}_{\partial B_j} = \inf\{t > \tau_{Q_j} : \sigma_t \in \partial B_j\} \tag{3.55}$$

and putting

$$\bar{\bar{\mathscr{F}}} = (\bar{\bar{\mathscr{F}}}^{1,u}; \bar{\bar{\mathscr{F}}}^{1,d}_{t_d^1}; \bar{\bar{\mathscr{F}}}^{2,u};...; \bar{\bar{\mathscr{F}}}^{N-1,d}_{t_d^{N-1}}; \bar{\bar{\mathscr{F}}}^{N,u}) \tag{3.56}$$

we get

$$\begin{aligned}
&(\bar{\bar{\mathscr{F}}}^{1,u}_{t_u^1};...; \bar{\bar{\mathscr{F}}}^{N-1,d}_{t_d^{N-1}}; \bar{\bar{\mathscr{F}}}^{N,u}_{t_u^N}) \\
&\quad = (\bar{\bar{\mathscr{F}}}^{1,u}; \bar{\bar{\mathscr{F}}}^{1,d}_{t_d^1}; \bar{\bar{\mathscr{F}}}^{2,u};...; \bar{\bar{\mathscr{F}}}^{N-1,d}_{t_d^{N-1}}; \bar{\bar{\mathscr{F}}}^{N,u}) \cap \mathscr{G} \\
&\quad = \left[\bigcup_{t_u^1=1}^{\infty} \bigcup_{t_d^1=1}^{t_d^1} \bigcup_{t_u^2=1}^{\infty} \cdots \bigcup_{t_d^{N-1}=1}^{t_d^{N-1}} \bigcup_{t_u^N=1}^{\infty} (\mathscr{F}^{1,u}_{t_u^1};...; \mathscr{F}^{N,u}_{t_u^N}) \right] \cap \mathscr{G} \\
&\quad \equiv \bar{\bar{\mathscr{F}}} \cap \mathscr{G}
\end{aligned} \tag{3.57}$$

To get a lower bound on the probability of $\bar{\bar{\mathscr{F}}} \cap \mathscr{G}$ one can use the lower bound

$$P(\bar{\bar{\mathscr{F}}} \cap \mathscr{G}) \geq P(\bar{\bar{\mathscr{F}}}) - P(\mathscr{G}^c) \tag{3.58}$$

Thus, if we get a bound of the form

$$P(\mathscr{G}^c) < a \leq \tfrac{1}{2} P(\bar{\bar{\mathscr{F}}}) \tag{3.59}$$

we would have

$$P(\bar{\bar{\mathscr{F}}} \cap \mathscr{G}) > \tfrac{1}{2} P(\bar{\bar{\mathscr{F}}}) \tag{3.60}$$

To evaluate the probability of the event $\bar{\bar{\mathscr{F}}}$, we can use the following general strategy. First, we evaluate the probability $P(\xi^j_{v_j} = S_{j+1})$ relative to the Markov chain ξ^j [see Eqs. (3.37) and (3.38)]. We have

$$P_{Q_j}(\xi^j_{v_j} = S_j) = \sum_{n=0}^{\infty} [1 - P(Q_j \to \partial B_j)]^n P(Q_j \to S_{j+1}) \tag{3.61}$$

From now on we suppose that the configurations $S_{i+1} \in \partial B_i \cap \partial B_{i+1}$, $i = 1,..., N-1$, $S_1 \in \partial B_1$, and $S_{N+1} \in \partial B_N$ are chosen so that

$$\min_{\sigma \in \partial B_j} H(\sigma) = H(S_j), \qquad j = 1,..., N \tag{3.62}$$

Using (3.41), we can thus infer that, for every $\varepsilon > 0$ and β sufficiently large,

$$P(Q_j \to \partial B_j) \leqslant \exp\{-\beta[H(S_j) - H(Q_j) - \varepsilon]\} \tag{3.63}$$

Let us suppose further that

$$H(S_j) > H(S_{j-1}) \tag{3.64}$$

and that we have the lower bounds

$$P(Q_j \to S_{j+1}) \geqslant \exp\{-\beta[H(S_{j+1}) - H(Q_j) + \varepsilon]\} \tag{3.65}$$

and

$$P_{S_{j+1}}(\theta_{j+1} \leqslant \bar{t}_d^j) \geqslant \exp(-\varepsilon\beta) \tag{3.66}$$

for every $\varepsilon > 0$ and β sufficiently large. Combining the bounds (3.63)–(3.65), we get a lower bound (3.61). Using now this bound together with definitions (3.50), (3.53), and (3.56), Eq. (3.51), and the bound (3.63), we conclude that

$$P(\bar{\mathscr{F}}) \geqslant \exp\{-\beta[H(S_N) - H(S_1) + \varepsilon]\} \tag{3.67}$$

for every $\varepsilon > 0$ and β sufficiently large. In our particular case, Eq. (3.62) and the inequalities (3.64) and (3.65) will be satisfied and we will get estimates of the form (3.66). On the other hand, we get upper bounds on $P(\mathscr{G}^c)$ that are superexponentially small in β [see the condition (C1) in the proof of Proposition 1] yielding (3.59) for β sufficiently large. Thus we get

$$P(\bar{\mathscr{F}} \cap \mathscr{G}) > \exp\{-\beta[H(S_N) - H(S_1) + \varepsilon]\} \tag{3.68}$$

for every $\varepsilon > 0$ and β sufficiently large.

Finally, to state Proposition 1, we need the definition of yet another stopping time. Namely, considering standard octagons $Q(D_1, D_2)$ with $D_i \geqslant 3l^* - 2$, $i = 1, 2$, we take $\bar{\tau}$ to be the first hitting time to a standard octagon different from $Q(D_1, D_2)$,

$$\bar{\tau} = \tau_{\mathfrak{S}_Q \setminus Q(D_1, D_2)} \tag{3.69}$$

Here

$$\mathfrak{S}_Q = \bigcup_{D_1, D_2} Q(D_1, D_2) \tag{3.70}$$

is the set of all standard octagons. Supposing that $D_2 \leqslant D_1$ with $3l^* - 2 < D_2 < D^*$, we shall use $E(L_2)$ as shorthand for $E(D_1, D_2)$ defined in (3.14):

$$\begin{aligned} E(L_2) &= 2[h(l^* - 1) - (2K - h)] + h(L_2 - 3) \equiv h(D_2 - 1) - 4K \\ &= h(L_2 - 1) - 2h\eta \end{aligned} \tag{3.71}$$

for $L_2 > l^* + 1$ and

$$E(l^* + 1) = 2h(l^* - 1) - (2K - h) \tag{3.72}$$

for $L_2 = l^* + 1$.

Proposition 1. Consider a standard octagon $Q(D_1, D_2)$ with $D_2 \leqslant D_1$ and let

$$3l^* - 2 < D_2 < D^* \tag{3.73}$$

Let, further,

$$\tau = \tau_{Q(D_1, D_2 - 1) \cup Q(D_1 - 1, D_2)} \qquad \text{if} \quad D_1 = D_2 \tag{3.74}$$

and

$$\tau = \tau_{Q(D_1 - 1, D_2)} \qquad \text{if} \quad D_2 < D_1 \tag{3.75}$$

Then, for every $\varepsilon > 0$, one has

$$\sup_{\sigma \in Q(D_1, D_2)} P_\sigma(\tau < \exp\{\beta[E(L_2) + \varepsilon]\} \text{ and } \tau = \bar{\tau}) \xrightarrow[\beta \to \infty]{} 1 \tag{3.76}$$

Proof. We suppose that $D_1 > D_2$. The case $D_1 = D_2$ does not present any supplementary difficulty and is left to the reader.

First, let us consider the case $L_2 > l^* + 1$. The main step in the proof will be to show that

$$P_\sigma(\tau_{\partial \mathcal{S}(D_1, D_2)} > \exp\{\beta[E(L_2) + \varepsilon]\}) \xrightarrow[\beta \to \infty]{} 0 \tag{3.77}$$

and

$$P_\sigma(\tau_Q > \exp\{\beta(2K - h + \varepsilon)\}) \xrightarrow[\beta\to\infty]{} 0 \tag{3.78}$$

for every $\varepsilon > 0$ and $\sigma \in \mathscr{D}(D_1, D_2)$. Once (3.77) and (3.78) are proven, we can reason in the following way.

From (3.77) we know, in particular, that starting from Q one reaches with high probability the set $\partial\mathscr{D}(D_1, D_2)$ before the time $\exp\{\beta[E(L_2)+\varepsilon]\}$. Moreover, it is improbable to reach $\partial\mathscr{D}(D_1, D_2)$ outside the set $\mathscr{S}(D_1, D_2)$ of configurations yielding minima of H on $\partial\mathscr{D}(D_1, D_2)$ [see the definitions (3.10) and (3.11)]. Indeed, one clearly has

$$\begin{aligned} P_Q(\tau_{\partial\mathscr{D}\setminus\mathscr{S}} < \tau_{\partial\mathscr{S}}) \leqslant{} & P_Q(\tau_{\partial\mathscr{D}\setminus\mathscr{S}} < \exp\{\beta[E(L_2)+\varepsilon]\}) \\ & + P_Q(\tau_{\partial\mathscr{D}} \geqslant \exp\{\beta[E(L_2)+\varepsilon]\}) \end{aligned} \tag{3.79}$$

for every $\varepsilon > 0$ (we have dropped the explicit dependence on D_1, D_2). The first term on the right-hand side can be bounded with the help of Lemma 3.4, taking into account that according to Lemma 3.3, for every $\bar\sigma \in \mathscr{S}(D_1, D_2)$ one has

$$H(\bar\sigma) = \min_{\sigma\in\partial\mathscr{D}(D_1, D_2)} H(\sigma) = H(Q(D_1, D_2)) + E(L_2) \tag{3.80}$$

Once reaching $\mathscr{S}(D_1, D_2)$, with high probability one of two possibilities occurs. Either we descend to $Q(D_1 - 1, D_2)$ with a fixed nonvanishing probability, or we return to $\mathscr{D}(D_1, D_2)$. The saddles in $\mathscr{S}$ consist just of a contracted octagon united with a unit-square protuberance and, with probability approaching 1 as $\beta \to \infty$, this protuberance is either cut off or is made stable by a flip of a minus spin adjacent to it. Actually, following the argument of the proof of Lemma 3.3, we can show that for every $\bar\sigma \in \mathscr{S}(D_1, D_2)$ one has

$$P_{\bar\sigma}(\tau = 1) > \frac{1}{|\Lambda|} \quad\text{and}\quad P_{\bar\sigma}(\tau_{\mathscr{S}(D_1,D_2)} < \tau_{\mathscr{S}^c(D_1,D_2)\setminus\{\bar\sigma\}} \mid \tau > 1) \xrightarrow[\beta\to\infty]{} 1 \tag{3.81}$$

After returning to $\mathscr{D}(D_1, D_2)$ and reaching Q, according to (3.78), we can repeat the attempt and prove, finally, with the help of the strong Markov property, that for every $\sigma \in \mathscr{D}(D_1, D_2)$ one has

$$P_\sigma(\tau < \exp\{\beta[E(L_2)+\varepsilon]\}) \xrightarrow[\beta\to\infty]{} 1 \tag{3.82}$$

To prove that

$$P_Q(\tau = \bar\tau) \xrightarrow[\beta\to\infty]{} 1 \tag{3.83}$$

for $\bar{\tau}$ defined in (3.69), or, in other terms, that

$$P_Q(\tau < \bar{\tau}) \xrightarrow[\beta \to \infty]{} 0 \tag{3.84}$$

we observe that, since $L_2 < L^*$, one has

$$\min_{\omega: Q(D_1, D_2) \to \mathfrak{S}_Q \setminus \{Q(D_1, D_2), Q(D_1 - 1, D_2)\}} \max_{\sigma \in \omega} [H(\sigma) - H(Q(D_1, D_2))] > E(L_2) \tag{3.85}$$

Thus, for all sufficiently small $\varepsilon > 0$, the set

$$\bigcup_{D'_1, D'_2 \neq (D_1, D_2), (D_1 - 1, D_2)} \mathscr{D}(D'_1, D'_2)$$

cannot be reached in a time $\exp\{\beta(E(L_2) + \varepsilon)\}$ with probability approaching one as $\beta \to \infty$.

For $L_2 = l^* + 1$, similar reasoning is slightly more complicated. Instead of (3.81) we introduce the set

$$\begin{aligned} \mathscr{C} = {} & \mathscr{D}(D_1 - 1, D_2) \\ & \cup \{Q\colon (D_1(Q), D_2(Q)) = (D_1, D_2) \text{ and } \min(L_1(Q), L_2(Q)) < l^*\} \end{aligned} \tag{3.86}$$

for which, for every $\bar{\sigma} \in \mathscr{S}(D_1, D_2)$, one has

$$P_{\bar{\sigma}}(\tau_{\mathscr{S}(D_1, D_2)} < \tau_{\mathscr{S}^c(D_1, D_2) \setminus \{\bar{\sigma}\}} \mid \tau_{\mathscr{C}} > 1) \xrightarrow[\beta \to \infty]{} 1 \tag{3.87}$$

Moreover (still $L_2 = l^* + 1$) we will show that

$$P_{\bar{\sigma}}(\tau < \exp\{\beta h(l^* - 2 + \varepsilon)\} \mid \tau_{\mathscr{C}} = 1) \xrightarrow[\beta \to \infty]{} 1 \tag{3.88}$$

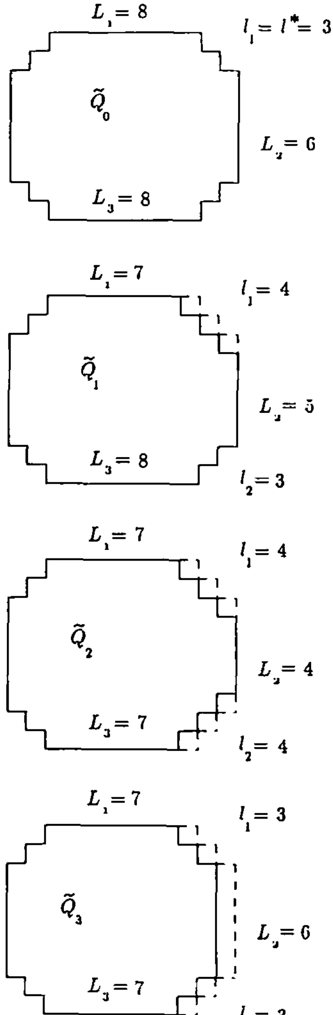
for any $\bar{\sigma} \in \mathscr{S}(D_1, D_2)$ and $\varepsilon > 0$. Thus, noticing that

$$h(l^* - 2) < E(l^* + 1)$$

we can conclude that (3.82) holds true.

Hence, to prove the proposition we are left with the proof of (3.77), (3.78), and for $L_2 = l^* + 1$ also (3.88). Actually, the crucial point is to get (3.77). The claim (3.78) will be a byproduct of the proof of (3.77) [see the condition (C3) below].

To derive the bound (3.77), we will look more carefully at a possible way of reaching the set $\partial\mathscr{D}$ from a configuration $\sigma \in \mathscr{D}$. The trajectory will be characterized by visiting several special octagons (see Scheme 3.8): First, we use $\tilde{Q}_0 \in Q(D_1, D_2, l^*, l^*, l^*, l^*)$ to denote the standard octagon



Scheme 3.8

representing our initial condition. We can suppose, without loss of generality, that it is centered—its upper left corner is the point $(-1/2, 1/2)$ of the dual lattice. As usual, we use $R(D_1, D_2)$ to denote the rectangle circumscribed to $\tilde{Q}_0$. The octagon $\tilde{Q}_1$ is contained in $\tilde{Q}_0$; it has the same rectangular envelope $R(D_1, D_2)$ and it differs from $\tilde{Q}_0$ only the length of one of its oblique edges—namely, the first one in the clockwise enumeration l_i, $i=1, 2, 3, 4$, starting with the uppermost right one is of length l^*+1, $l_1=l^*+1$. The length of all the remaining oblique edges is again l^*,

$$\tilde{Q}_1 \in Q(D_1, D_2, (l_i)_{i=1,\dots,4}, l_1=l^*+1, l_2=l_3=l_4=l^*)$$

Further, the octagon $\tilde{Q}_2$ is obtained from $\tilde{Q}_1$ by replacing the oblique edge l_2 (of length l^*) adjacent to the shortest coordinate edge by an edge of length l^*+1. The octagon $\tilde{Q}_2$ is thus again centered and has two oblique edges of length l^*+1 adjacent to the shortest coordinate edge. Finally, the octagon $\tilde{Q}_3$ is the element of $Q(D_1, D_2-1, l^*, l^*, l^*, l^*)$ obtained from $\tilde{Q}_2$ by erasing the shortest coordinate edge (the vertical one on the right-hand side) adjacent to the oblique edges of length l^*+1. The saddle configurations $\tilde{S}_i$, $i=1, 2, 3$, are obtained from $\tilde{Q}_i$ by adding a unit-square protuberance (the first one in lexicographic order—denoted by a dot in Scheme 3.8) to (1) the first (uppermost left) oblique edge in the case of $\tilde{Q}_1$; (2) the second oblique edge (down on the right) in the case of $\tilde{Q}_2$; and (3) one of the shortest coordinate edges—namely, to the vertical one on the right-hand side—in the case of $\tilde{Q}_3$.

Let us first consider the case $L_2 > l^*+1$.

To prove (3.77), we follow ref. 8 and introduce an event $\mathscr{E}^s_\sigma$ (of shrinking) starting from an arbitrary σ in $\mathscr{D}(D_1, D_2)$, taking place in an interval of time $T_1 = T_1(\delta_1)$,

$$T_1(\delta_1) = \exp\{\beta(2K - h + \delta_1)\} \tag{3.89}$$

with sufficiently small $\delta_1 > 0$ to be specified later, and such that:

1. If $\mathscr{E}^s_\sigma$ takes place, then necessarily the set $\partial\mathscr{D}$ is reached (in a particular manner) before the time $T_1(\delta_1)$.
2. The probability $P(\mathscr{E}^s_\sigma)$ satisfies the uniform lower bound

$$\inf_{\sigma \in \mathscr{D}(D_1, D_2)} P(\mathscr{E}^s_\sigma) \geqslant \alpha \tag{3.90}$$

with α chosen large enough to satisfy

$$\lim_{\beta \to \infty} (1-\alpha)^{T_1/T_2} = 0 \tag{3.91}$$

Here $T_2 = T_2(\delta_2)$ (again, the constant $\delta_2 > 0$ will be specified later) is given by

$$T_2(\delta_2) = \exp\{\beta[E(L_2) + \delta_2]\} \tag{3.92}$$

Splitting now the interval T_2 into T_2/T_1 intervals of length T_1, we can argue that the probability that, during any such interval, one has not reached $\partial\mathscr{D}$ is at most $1-\alpha$, since not reaching $\partial\mathscr{D}$ in time T_1 means that $\mathscr{E}^s_\sigma$ certainly did not take place. Using now the strong Markov property and taking into account (3.91), we see that an attempt to realize the event $\mathscr{E}^s_\sigma$ not later than T_2 will be successful with high probability for large β—the equality (3.77) follows.

Now, let us first describe the event $\mathscr{E}^s_\sigma$ of shrinking in words; a formal definition will follow.

Starting from any σ in $\mathscr{D}(D_1, D_2)$, we first descend to $\tilde{Q}_0$. Then we stay for a time of the order $\exp\{\beta(2K-h)\}$ inside $\mathscr{B}(\tilde{Q}_0)$, the basin of attraction of $\tilde{Q}_0$, leaving it afterward through the saddle $\tilde{S}_1$. After reaching $\tilde{S}_1$ we descend to $\tilde{Q}_1$ in a unique step. Staying inside $\mathscr{B}(\tilde{Q}_1)$ for a time of order $\exp\{\beta(2K-h)\}$, we leave through $\tilde{S}_2$, from which, again in one step, we descend to $\tilde{Q}_2$. Finally, again staying in $\mathscr{B}(\tilde{Q}_2)$ for a time of order $\exp(2K-h)\}$, we ascend to $\tilde{S}_3 \in \mathscr{S}(D_1, D_2)$.

Let us turn to a formal definition of $\mathscr{E}^s_\sigma$.

For every σ in $\mathscr{D}(D_1, D_2)$ and every $t_0 \in \mathbb{N}$, let

$$\mathscr{E}^0_{\sigma, t_0} = \{\sigma_0 = \sigma, \tau_{\tilde{Q}_0} = t_0\} \tag{3.93}$$

Namely, $\mathscr{E}^0_{\sigma, t_0}$ describes the event, after starting from σ, of hitting at the moment t_0, for the first time, the octagon $\tilde{Q}_0$. Now, consider the events $\mathscr{F}^{1,u}_{t^1_u}$, $\mathscr{F}^{1,d}_{t^1_d}$, $\mathscr{F}^{2,u}_{t^2_u}$, $\mathscr{F}^{2,d}_{t^2_d}$, and $\mathscr{F}^{3,u}_{t^3_u}$, defined by (3.47)–(3.50) with $N=3$; $Q_1, Q_2, Q_3 = \tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2$; $B_1, B_2, B_3 = \mathscr{B}(\tilde{Q}_0), \mathscr{B}(\tilde{Q}_1), \mathscr{B}(\tilde{Q}_2)$; and $S_2, S_3, S_4 = \tilde{S}_1, \tilde{S}_2, \tilde{S}_3$. Moreover, we take $\bar{t}^1_u = \bar{t}^2_u = \bar{t}^3_u = \exp\{\beta(2K-h+\delta)\}$ and $\bar{t}^1_d = \bar{t}^2_d = 1$. Further, we consider the time $\bar{t}_0 = \exp\{\beta(2K-h+\delta)\} \equiv T_1(\delta)$ ($\delta > 0$, sufficiently small, to be specified later) for the first descent from $\partial\mathscr{D}(D_1, D_2)$ to $\tilde{Q}_0$. Finally, the configuration S_1 is one of the minimal saddles in $\partial\mathscr{B}(\tilde{Q}_0)$ satisfying the equality

$$\min_{\sigma \in \partial\mathscr{B}(\tilde{Q}_0)} H(\sigma) = H(S_1)$$

It is obtained by adding a unit-square protuberance to one of the oblique edges of $\tilde{Q}_0$. Clearly,

$$H(S_1) = H(\tilde{Q}_0) + 2K - h \tag{3.94}$$

Introducing now

$$\mathscr{E}^s_\sigma = (\bar{\mathscr{E}}^0_{\sigma, \bar{t}^0}; \bar{\mathscr{F}}^{1,u}; \mathscr{F}^{1,d}_1; \bar{\mathscr{F}}^{2,u}; \mathscr{F}^{2,d}_1 \bar{\mathscr{F}}^{3,u}) \cap \mathscr{G} \tag{3.95}$$

[cf. (3.57)], we will suppose that the following conditions are satisfied:

(C1) $P(\mathscr{G}^c) \leqslant \exp(-e^{C\beta})$ for a constant $C > 0$.

(C2) $P(\bar{\mathscr{F}}_1^{j,d}) > \lambda$, $j = 1, 2$, for a constant $\lambda > 0$ independent of β.

(C3) $P(\bar{\mathscr{E}}^0_{\sigma, t_0}) = \sum_{t_0=1}^{\bar{t}_0} P(\mathscr{E}^0_{\sigma, t_0}) \geqslant \exp(-\varepsilon\beta)$ for every $\varepsilon > 0$ and all β sufficiently large.

Let us notice that the equality (3.62) and the inequality (3.64) are satisfied in our case, that the validity of (3.66) immediately follows from an explicit computation, based on the definition of Metropolis dynamics, taking into account that the transitions $\tilde{S}_{j+1} \to \tilde{Q}_j$, $j = 1, 2, 3$, are just single-spin-flip events, and, finally, that the validity of (3.65) is immediate by considering a particular event leading from Q_j to S_{j+1} consisting of a set of subsequent h-erosions in the sites adjacent from the interior to the concerned oblique or vertical edge of $\tilde{Q}_0$, $\tilde{Q}_1$, and $\tilde{Q}_2$ (see Scheme 3.8). [The same argument actually proves also the stronger condition (C2).]

From the strong Markov property we get

$$P(\mathscr{E}^s_\sigma) = P(\bar{\mathscr{E}}^0_{\sigma, t^0})\, P[(\bar{\mathscr{F}}^{1,u}; \mathscr{F}_1^{1,d}; \mathscr{F}^{2,u}; \mathscr{F}_1^{2,d}; \bar{\mathscr{F}}^{3,u}) \cap \mathscr{G}] \tag{3.96}$$

Using (3.66) together with (3.65), we can infer, as we have seen, the lower bound (3.67). From (C1) and (3.67), for β sufficiently large, we get (3.59), (3.60), and thus also (3.68). Finally from (C3), (3.68), (3.94), and (3.96) we deduce that

$$\inf_{\sigma \in \mathscr{S}(D_1, D_2)} P(\mathscr{E}^s_\sigma) \geqslant \alpha \equiv \exp\{-\beta[E(L_2) - 2K + h + \varepsilon]\} \tag{3.97}$$

for every $\varepsilon > 0$ and β sufficiently large. The validity of (3.91) the follows from the bound (3.97)— for any fixed δ_2 [see Eq. (3.92)] once we take, say, $\delta = \delta_1/2 = \delta_2/4$ for δ from the definition of the times $\bar{t}^i_u$.

Hence, to conclude the proof of (3.77), we only need to verify the conditions (C1)–(C3) above.

To get the bound (C1), we suppose that for every $j = 0, 1, 2$ there exist a time $\bar{t}^j_u$ and for every $\sigma \in \mathscr{B}(\tilde{Q}_j)$ an event $\hat{\mathscr{E}}^j_\sigma$ of escape from $\mathscr{B}(\tilde{Q}_j)$ such that:

(i) The occurrence of $\hat{\mathscr{E}}^j_\sigma$ implies that

$$\tau_{\partial\mathscr{B}(\tilde{Q}_j)} < \tilde{t}^j_u \tag{3.98}$$

(ii) For every $\varepsilon_0 > 0$ sufficiently small and for every β sufficiently large,

$$\inf_{\sigma \in B_j} P(\hat{\mathscr{E}}^j_\sigma) \geqslant \frac{\tilde{t}^j_u}{\bar{t}^j_u} \exp(\varepsilon_0 \beta) \tag{3.99}$$

The superexponential estimate (C1) then directly follows from (3.98) and (3.99) with the help of the strong Markov property.

To construct the event $\hat{\mathscr{E}}^j_\sigma$, we repeat, in the present simpler situation of escape from $\mathscr{B}=\mathscr{B}(\tilde{Q}_j)$, a construction similar to the one discussed above in the case of escape from $\mathscr{D}$. Namely, we choose time $\tilde{t}^j_u = T_0 + 1$ with

$$T_0 = (8\tilde{J} + 8K + 2h)\,|\Lambda|/h \tag{3.100}$$

which is the time that suffices, starting from any configuration, to reach a local minimum [see Eq. (3.5)], and set

$$\hat{\mathscr{E}}^j_\sigma = (\hat{\mathscr{E}}^0_{\sigma,T_0}; \hat{\mathscr{F}}^{j,u}) \tag{3.101}$$

where

$$\hat{\mathscr{F}}^{j,u} = \{\xi^j_0 = \tilde{Q}_j, \xi^j_{v_j} = \tilde{S}_j, v_{v_j} = 1\} \tag{3.102}$$

and

$$\hat{\mathscr{E}}^0_{\sigma,T_0} = \{\sigma_0 = \sigma, \tau_{\tilde{Q}_j} < T_0, \sigma_t = \tilde{Q}_j \text{ for all } t \in [\tau_{\tilde{Q}_j}, T_0]\} \tag{3.103}$$

The saddle configurations $\tilde{S}_j$ for $j = 1, 2$ (and 3) have already been defined. The configuration $\tilde{S}_0$ is identical to S_1 introduced above and it is obtained by adding a unit-square protuberance to one of the oblique edges of $\tilde{Q}_0$.

The event $\hat{\mathscr{E}}^j_\sigma$ thus describes, starting from a generic $\sigma \in \mathscr{B}(\tilde{Q}_j)$, a descent to $\tilde{Q}_j$ in a time shorter than T_0 and staying in $\tilde{Q}_j$ up to T_0. The event $\hat{\mathscr{F}}^{j,u}$ simply consists of creating of a K-protuberance on an oblique edge of $\tilde{Q}_j$.

It is easy to see that, given δ (see definition of $\tilde{t}^j_u$), the bound (3.99) is satisfied provided $\varepsilon_0 < \delta/4$.

As we already mentioned, the condition (C2) follows from the definition of $\tilde{S}_1$ and $\tilde{S}_2$ with $\lambda = 1/|\Lambda|$. With this probability the spin chosen for updating is the one on the unit-square protuberance of $\tilde{S}_1$, $\tilde{S}_2$, respectively.

The idea of the proof of the condition (C3) is as follows. Starting from any $\sigma \in \mathscr{D}(D_1, D_2)$, after a time of order T_0, (3.100), we descend to the octagon Q in $\mathscr{D}(D_1, D_2)$ with high probability. Then, in a time of order $\exp\{\beta(2K - h + \tilde{\varepsilon})\}$, with $\tilde{\varepsilon} > 0$ such that

$$2K - h + \tilde{\varepsilon} < h(l^* - 1) - \tilde{\varepsilon} \tag{3.104}$$

many K-protuberances occur with high probability, but no total erosion of any oblique edge of length $l \geqslant l^*$ takes place.

To provide a formal proof we introduce again a Markov chain—this time the chain $\{\eta_n\}$ obtained by looking at our process σ_t at the times of passing through octagons. The space of states of our chain is the set Y of all octagons Q in Λ,

$$Y = \{Q; Q \subset \Lambda\}$$

Let

$$\bar{v}_0 = 0, \qquad \bar{u}_n = \inf\{t > \bar{v}_n, \bar{\sigma}_t \notin Y\}$$
$$\bar{v}_{n+1} = \inf\{t > \bar{u}_n : \bar{\sigma}_t \in Y\}$$

We set

$$\eta_n = \sigma_{\bar{v}_n}$$

and, for every $A \subset Y$, introduce

$$\bar{v}_A = \inf\{n; \eta_n \in A\} \tag{3.105}$$

Let

$$\bar{Y} = \{Q: (D_1(Q), D_2(Q)) = (D_1, D_2)\} \supset \{Q \in \mathscr{D}(D_1, D_2)\} \tag{3.106}$$

and

$$Y^< = \{Q \equiv Q(D_1, D_2, (l_i)_{i=1,\dots,4}) \in \bar{Y}: l_i \leqslant l^*, i = 1,\dots, 4\} \tag{3.107}$$

We will prove that for all $\varepsilon > 0$ one has

$$\sup_{Q \in Y} P_Q(\bar{v}_{Y^<} > \exp\{\beta(2K - h + \varepsilon)\}) \xrightarrow[\beta \to \infty]{} 0 \tag{3.108}$$

Postponing the proof of (3.108), let us first show that for every $\sigma \in \mathscr{D}(D_1, D_2)$, $\varepsilon > 0$, and β sufficiently large,

$$P_\sigma(\tau_Y < \exp(\varepsilon\beta)) > 1/2 \tag{3.109}$$

This implies, with the help of the strong Markov property, that for every $\varepsilon > 0$, $\sigma \in \mathscr{D}(D_1, D_2)$, and β sufficiently large, one has

$$P_\sigma(\tau_Y > \exp(2\varepsilon\beta)) < (1/2)^{\exp(\varepsilon\beta)} \tag{3.110}$$

To get the lower bound (3.109), one again uses the property of time T_0—with a strictly positive probability independent of β, starting from any configuration one reaches a local minimum in a time at most T_0.

From (3.110) one gets, for every $A \subset \bar{Y}$,

$$P_Q(\tau_A < \exp(\varepsilon\beta)\, v_A) \xrightarrow[\beta \to \infty]{} 1 \tag{3.111}$$

Hence, by (3.111) and (3.108) we get

$$P_Q(\tau_{Y^<} < \exp\{\beta(2K-h+\varepsilon)\}) \xrightarrow[\beta\to\infty]{} 1 \tag{3.112}$$

for every $\varepsilon > 0$. On the other hand, with the same reasoning as that leading to (3.110), we obtain

$$\inf_{\sigma\in\mathscr{S}(D_1,D_2)} P_\sigma(\tau_{\bar{Y}} < T_0) > \exp(-\varepsilon\beta) \tag{3.113}$$

for every $\varepsilon > 0$ and β sufficiently large.

Now, let us prove (3.108). This can be obtained very easily, once we take ε in (3.108) sufficiently small—namely, $\varepsilon = \bar{\varepsilon}$ satisfying the inequality (3.104). For every $Q = Q(D_1, D_2, (l_i)_{i=1,...,4})$ such that for a nonempty subset of indices, $J \subset \{1,..., 4\}$, one has $l_j \geqslant l^*$, $j \in J$, let $Y^<(Q)$ be the set

$$Y^<(Q) = \{Q' \equiv Q'(D_1, D_2, (l_i')_{i=1,...,4}): l_j' \leqslant l_j \text{ for all } j \in J\} \tag{3.114}$$

We have

$$P(Q \to Y\backslash(Y^<(Q) \cup \{Q\})) = P_Q(\sigma_{\bar{\varepsilon}_1} \notin Y^<(Q)) \leqslant \exp\{-\beta[h(l^*-1) - \varepsilon]\} \tag{3.115}$$

for every $\varepsilon > 0$ and β sufficiently large. Indeed, the bound (3.115) follows easily by reversibility, in a similar way as Eq. (3.41), since for every $Q \in \mathscr{D}(D_1, D_2)$ one has

$$\min_{\omega: Q\to Y\backslash(Y^<(Q)\cup\{Q\})} \max_{\sigma\in\omega} H(\sigma) \geqslant H(Q) + h(l^*-1)\,\beta \tag{3.116}$$

On the other hand, one easily sees that for every $\varepsilon > 0$ and β sufficiently large,

$$P(Q \to Y\backslash\{Q\}) \equiv P_Q(\sigma_{\bar{\varepsilon}_1} \neq Q) \geqslant \exp\{-\beta(2K-h+\varepsilon)\}$$

and, moreover,

$$P(Q \to Y^<(Q)\backslash\{Q\}) \geqslant \exp\{-\beta(2K-h+\varepsilon)\}$$

It is easy to see that, starting from any configuration $\sigma \in Y^<$ we reach the minimum $\tilde{Q}_0$, with high probability, in a time of order $\exp\{\beta h(l^*-2)\}$. Indeed, in a time of the order

$$T_1 = \exp\{\beta[h(l^*-2) + \varepsilon_1]\} \tag{3.117}$$

if ε_1 is sufficiently small, no K-protuberances (or K-erosions) or *a fortiori* elementary events involving bigger increments of energy take place with probability larger than $\exp(-\varepsilon_1 \beta)$ for β sufficiently large. Hence, with the same probability, the only possible elementary events are h-erosions and recoveries. Taking this fact into account, for all $\sigma \in Y^<$, we get

$$P_\sigma(\tau_{\tilde{Q}_0} < T_1) > e^{-\beta\varepsilon} \tag{3.118}$$

The proof of (3.118) can be obtained by adapting the argument of proof of Theorem 1(a) of ref. 11.

We only present here the main idea leading to (3.118). Starting from any $\sigma \in \mathscr{B}(\tilde{Q}_0)$, there is a probability larger than $\exp\{-\beta[h(l^*-2)-\varepsilon]\}$ to completely erode, in a time l^*-1, any oblique edge of $\tilde{Q}_0$ of the length $\leqslant l^*-1$. Hence, in a time of order T_1, we certainly have to reach $\tilde{Q}_0$ before T_1 since the circumscribed octagon cannot grow and since *a fortiori* we know from the inequality

$$h(l^*-1) > (2K-h)$$

that no oblique edge $l \geqslant l^*$ can be completely eroded.

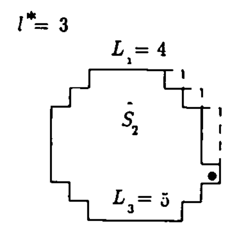
From (3.112), (3.113), and (3.118) we get (C3). Namely, for every $\varepsilon > 0$ one has

$$P(\bar{\mathscr{E}}^0_{\sigma, \bar{t}_0}) = \sum_{t_0=1}^{\bar{t}_0} P(\mathscr{E}^0_{\sigma, t_0}) \geqslant \exp(-\varepsilon\beta) \tag{3.119}$$

This concludes the proof of Proposition 1 when $L_2 > l^*+1$.

Consider now the case $L_2 = l^*+1$ (supposing always $L_1 > L_2$). The proof of Proposition 1 in this case is similar to the one in the case $L_2 > l^*+1$. For any $\sigma \in \mathscr{D}(D_1, D_2)$ we will again introduce an event $\mathscr{E}^s_\sigma$. The main differences can be summarized in the following two points.

1. It follows from Lemma 3.3 that the minimum of H in $\partial\mathscr{D}(D_1, D_2)$ is not achieved in $\tilde{S}_3$ but in $\tilde{S}_2$ (cf. Scheme 3.8) and in $\hat{S}_2$ defined as the saddle configuration obtained from $\tilde{Q}_1$ by eroding the last $L_2 - 2 = l^* - 1$ unit squares adjacent to the coordinate edge of length $L_2 - 1 = l^*$ [in other words, $\hat{S}_2$ is obtained by adding a unit-square protuberance to the coordinate edge of length l^*+2 of an octagon in $Q(D_1-1, D_2, l^*, l^*-1, l^*, l^*)$; see Scheme 3.9].



Scheme 3.9

2. The path starting from $\tilde{S}_2$ and reaching $\tilde{Q}_3$ is no longer a pure descent, but it involves tunneling. We have

$$H(\tilde{S}_2) - H(\tilde{Q}_0) = H(\hat{S}_2) - H(\tilde{Q}_0) = 2h(l^* - 1) - (2K - h) = E(l^* + 1)$$

and

$$H(\tilde{S}_2) = H(\hat{S}_2) > H(\tilde{S}_3) = H(\tilde{Q}_0) + 2h(l^* - 1) \\ - (2K - h) + h(l^* - 2) - (2K - h)$$

Let $D_1 > D_2 = l^* + 1 + 2(l^* - 1)$, i.e., $L_2 = l^* + 1$. The configurations $\tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2$ and $\tilde{S}_1$, $\tilde{S}_2$, $\tilde{S}_3$ are defined in exactly the same way as in the case $L_2 > l^* + 1$.

Again, we define for every $\sigma \in \mathscr{D}(D_1, D_2)$ the event $\mathscr{E}^0_{\sigma, t_0}$ [cf. (3.93)] in the same manner as before.

Now, let

$$Q_1, Q_2 = \tilde{Q}_0, \tilde{Q}_1; \qquad S_1, S_2 = \tilde{S}_1, \tilde{S}_2$$

and let $\bar{t}_0 = \exp\{\beta(2K - h + \delta)\}$, $\bar{t}^1_u = \bar{t}^2_u = \exp\{\beta(2K - h + \delta)\}$, and $t^1_d = 1$. We define

$$\mathscr{E}^s_\sigma = (\bar{\mathscr{E}}^s_{\sigma, t^0}; \bar{\mathscr{F}}^{1,u}; \mathscr{F}^{1,d}_1; \bar{\mathscr{F}}^{2,u}) \cap \mathscr{G} \tag{3.120}$$

[cf. (3.57) and (3.94)]. We get

$$\inf_{\sigma \in \mathscr{S}(D_1, D_2)} P(\mathscr{E}^s_\sigma) \geq \alpha \equiv \exp\{-\beta[E(l^* + 1) - 2K + h + \varepsilon]\} \tag{3.121}$$

One verifies (3.121) in the exactly same way as in the case $L_2 > l^* + 1$. We only have to check, in our case when $L_2 = l^* + 1$, the validity of the bound (3.119).

However, the proof of (3.119) is even simpler since now the number of possible cases to be considered, namely the number of Q's in $\bar{Y}$, is much smaller; in fact, $E(l^* + 1) < hl^*$ and then no oblique edge of length bigger than $l^* + 1$ can appear with high probability before the time $\exp\{\beta E(l^* + 1)\}$.

Finally, to conclude the proof of Proposition 1 we have to verify (3.88). We remark that entering into the set $\mathscr{C}$ [see Eq. (3.86)] from $\tilde{S}_2$ in one step means to descend to $\tilde{Q}_2$. The octagon $\tilde{Q}_2$ contains a vertical edge of length $l^* - 1$. Thus, to get (3.88), we reason in exactly the same way as when proving Eq. (3.118). We leave the details to the reader. This concludes the proof of Proposition 1. ■

In Proposition 1 we were considering the shrinking of a standard octagon with $D_1 \geqslant D_2 > 3l^* - 2$. Now we turn to the case $D_1 > D_2 = 3l^* - 2$. This means that $L_2 = l^*$ and

$$E(L_2) = E(l^*) = h(l^* - 1) \tag{3.122}$$

according to (3.14) and (2.18).

Proposition 2. Consider a standard octagon $Q(D_1, D_2)$ with

$$D_1 > D_2 = 3l^* - 2$$

Then, for all $\varepsilon > 0$, one has

$$P_{Q(D_1, D_2)}(\tau_{Q(D_1 - 1, D_2)} > \exp\{\beta[E(l^*) + \varepsilon]\}) \xrightarrow[\beta \to \infty]{} 0 \tag{3.123}$$

Proof. The proof is obtained along the same lines as in the case $L_2 = l^* + 1$ above. The main difference (and simplification) is that typical paths to the saddle configurations in $\mathscr{S}(D_1, D_2)$ are now purely uphill, while the paths from $\mathscr{S}(D_1, D_2)$ to $Q(D_1 - 1, D_2)$ involve two, instead of one, tunneling phenomena.

Let the octagons $\tilde{Q}_0$, $\tilde{Q}_1$, $\tilde{Q}_3$ and the saddle configuration $\tilde{S}_1$ be defined as for $L_2 \geqslant l^* + 1$. In particular $\tilde{S}_1$ is obtained from $\tilde{Q}_1 \in Q(D_1, D_2, l^* + 1, l^*, l^*, l^*)$ by adding a unit square to the oblique edge of length $l^* + 1$. Moreover, let S_1^* be the saddle configuration obtained from the centered octagon $\hat{Q}_1$ in $Q(D_1 - 1, D_2, l^* - 1, l^* - 1, l^*, l^*)$ by adding a unit square to its right-hand vertical edge of length $l^* + 2$.

We have $H(\tilde{S}_1) = H(S_1^*) = H(\tilde{Q}_0) + h(l^* - 1)$.

The minimum of H in $\partial\mathscr{D}(D_1, D_2)$ is now reached, again according to Lemma 3.3, in $\tilde{S}_1$ and S_1^*. We define $\mathscr{C}$ as in (3.86) and the shrinking event $\mathscr{E}_\sigma^s$ simply by

$$\mathscr{E}_\sigma^s = (\bar{\mathscr{E}}_{\sigma, \bar{t}_0}^0 ; \mathscr{F}^{1,u}) \cap \mathscr{G}$$

where the event $\bar{\mathscr{E}}_{\sigma, \bar{t}_0}^0$ is, for all $\sigma \in \mathscr{D}(D_1, D_2)$, defined in terms of events $\mathscr{E}_{\sigma, t_0}^0$, as in (3.44), while $\mathscr{E}_{\sigma, t_0}^0$ is defined as in (3.93); for $\bar{t}^0$ we take $\bar{t}^0 = \exp[h(l^* - 2) + \delta]$ and

$$\mathscr{F}^{1,u} = \{\xi_0^1 = \tilde{Q}_0, \xi_{v_1}^1 = \tilde{S}_1\}$$

$$\mathscr{G} = \{\tilde{\tau}_{\partial\mathscr{B}(\tilde{Q}_0)} \leqslant \bar{t}_u\}$$

$$\bar{t}_u = \exp\{\beta(2K - h + \delta)\}$$

Proposition 2 follows once we prove, in the present case, the bounds (3.88) and (3.119).

To get these two bounds we again argue as in the proof of (3.118) for the case $L_2 = l^* + 1$. In particular, we notice that now, entering into $\mathscr{C}$ in one step actually means to descend to $\tilde{Q}_1$. The octagon $\tilde{Q}_1$ itself now has a coordinate edge (the vertical one on the right-hand side) of length $l^* - 1$. Thus, in a time $t = \exp\{\beta[h(l^* - 2) + \delta]\}$, with δ so small that

$$h(l^* - 2) + \delta < (2K - h) - \delta$$

no K-protuberance (or elementary events involving even higher increment in energy) takes place with high probability and, as a consequence, the vertical edge of length $l^* - 1$ will be completely eroded. In this way we reach the octagon

$$Q_2^* \in Q(D_1 - 1, D_2, l^*, l^* - 1, l^*, l^*)$$

from which, in time $t = \exp\{\beta[h(l^* - 2) + \delta]\}$, with the help of the same mechanism, we descend to $\tilde{Q}_3$. We leave the details to the reader. ■

Propositions 3A–3C describe subsequent shrinking once we reach a regular octagon $Q(l^*)$. In the first step it shrinks by cutting one arbitrary (coordinate or oblique) edge.

Proposition 3A. Consider a regular octagon $Q(l^*)$. Let G_1 be the set of (not standard) octagons

$$Q(D_1, D_2, l^* - 1, l^*, l^*, l^* - 1), \qquad D_1 = 3l^* - 2, \qquad D_2 = 3l^* - 3$$

i.e.,

$$L_1 = l^* + 2, \qquad L_2 = L_3 = L_4 = l^* \quad \text{(modulo rotations)}$$

and G_2 be the set of octagons

$$Q(D_1, D_2, l^* + 1, l^*, l^*, l^*), \qquad D_1 = D_2 = 3l^* - 2$$

$$L_1 = l^* - 1 = L_2, \qquad L_3 = L_4 = l^* \quad \text{(modulo rotations)}$$

Then, for any $\varepsilon > 0$, one has

$$P_{Q(l^*)}(\tau_{G_1 \cup G_2} > \exp\{\beta[E(l^*) + \varepsilon]\}) \xrightarrow[\beta \to \infty]{} 0 \tag{3.124}$$

Proof. The proof is a straightforward adaptation of the proof of Proposition 2. ■

Thus, starting from $Q(l^*)$, we reach an octagon with at least one side shorter than l^*. This condition is maintained also during the subsequent shrinking and we get:

Proposition 3B. Consider an octagon Q with the values d_1, d_2, D_1, D_2 such that

$$\text{either} \quad \min(d_1, d_2) < 4l^* - 2 \qquad \text{or} \quad \min(D_1, D_2) < 3l^* - 2 \tag{3.125}$$

Then, for every $\varepsilon > 0$,

$$P_Q(\tau_{-\underline{1}} > \exp\{\beta[h(l^* - 2) + \varepsilon]\}) \xrightarrow[\beta \to \infty]{} 0 \tag{3.126}$$

Proof. We give only a sketch of the proof, leaving the details to the reader. (See also the discussion of the growth event $\mathscr{E}^{(r)}$ introduced in Section 5.) We first observe that by the hypothesis, one has initially at least one edge of length $l < l^*$. In a time of order $\exp\{\beta[h(l^* - 2) + \delta]\}$, with

$$h(l^* - 2) + \delta < (2K - h) - \delta$$

the only elementary processes taking place are h-erosions and recoveries (no K-protuberance or even more unlikely elementary events) with high probability, for large β. Hence, one of the minimal sides will be, with high probability for large β, eroded, as follows again from the arguments of the proof of Theorem 1(a) in ref. 11.

Now, after erosion of any of these edges, we pass to another octagon, with d_1, d_2, D_1, or D_2 decreased by one, in which there are still short ($\leqslant l^* - 1$) edges. Continuing this process, we finally reach the configuration $-\underline{1}$. ■

Actually, the process of shrinking after reaching $Q(l^*)$ (resp. $G_1 \cup G_2$) may be described in greater detail. It will be useful to consider at once a collection of steps leading from a regular octagon to another regular octagon of smaller size. To describe it, we will introduce several simple geometrical notions. First, we want to define a *cut* operation of an oblique or coordinate edge.

We say that we pass from Q to $Q' \subset Q$ via an *oblique cut* if Q' is obtained from Q by eliminating all the unit squares adjacent from the interior to an oblique edge of Q. As the result of an oblique cut the length of the cut oblique edge increases by one, whereas the lengths of the two coordinate adjacent edges decrease by one; d_1 or d_2 decreases by one.

For example, an oblique cut of a NW–SE edge with extremes y_1, x_2 in the octagon in Scheme 2.2 leads from d_1, d_2, l_1, l_2, l_3, l_4 to $d_1 - 1$, d_2, $l_1 + 1$, l_2, l_3, l_4. If we consider the corresponding values of the parameters

D_1, D_2, L_1, L_2, L_3, L_4 characterizing the same octagon, we pass to D_1, D_2, $L_1 - 1$, $L_2 - 1$, L_3, L_4.

We define also a coordinate (horizontal or vertical) cut on Q as the elimination of all the unit squares adjacent from the interior to a coordinate edge. As a result of a coordinate cut the corresponding length of the cut coordinate edge increases by two, whereas the two oblique adjacent edges decrease by one; the length D_1 or D_2 decreases by one.

For example, by a horizontal cut of the edge with extremes x_1, y_1 (see Scheme 2.2) we pass from D_1, D_2, L_1, L_2, L_3, L_4 to D_1, $D_2 - 1$, $L_1 + 2$, L_2, L_3, L_4, or, equivalently, from d_1, d_2, l_1, l_2, l_3, l_4 to d_1, d_2, $l_1 - 1$, l_2, l_3, $l_4 - 1$.

Since cuts of all sides will participate in the subsequent shrinking, it is useful to label all sides of an octagon Q in a unified way. Namely, we use $\mathscr{L}_i$, $i = 1,..., 8$, to denote the eight edges of Q by taking $\mathscr{L}_i = l_i$, $\mathscr{L}_{i+4} = L_i$, $i = 1,..., 4$.

We say that a cut of Q (oblique or coordinate) is *canonical* if it acts on an edge $\mathscr{L}_j$ of *minimal* length,

$$\mathscr{L}_j = \min_{i=1,...,8} \mathscr{L}_i$$

Lemma 3.5. Consider a regular octagon in $Q(l)$, $l > 2$. Suppose we apply to Q sequentially a series of arbitrary canonical cuts (i.e., we always arbitrarily choose the edge to cut among the ones of minimal length). Then, for every l and for any such sequence:

(i) After exactly 14 canonical cuts we always reach an element of $Q(l-1)$, namely, a regular octagon with the edge length decreased by one. We use $\mathscr{M}(l)$ to denote the set of all sequences of canonical contractions from $Q(l)$ to $Q(l-1)$.

(ii) Any sequence $M = \{Q^{(0)} \in Q(l), Q^{(1)},..., Q^{(13)}, Q^{(14)} \in Q(l-1)\} \in \mathscr{M}(l)$ contains only "almost regular" octagons in the sense that the difference in the lengths of any two oblique or coordinate edges in any $Q \in M$ is, for every $M \in \mathscr{M}(l)$, at most 3, $\max \mathscr{L}_i - \min \mathscr{L}_i \leqslant 3$. Moreover, the minimal possible length of an edge during any canonical sequence M is $l-2$, while the maximal one is $l+2$.

(iii) In any sequence $M \in \mathscr{M}(l)$ we perform exactly four oblique cuts in the NE–SW direction, four cuts in the NW–SE direction, three horizontal cuts, and three vertical cuts. There is one cut of an edge of length l, nine cuts of edges of length $l-1$, and four cuts of edges of length $l-2$.

Proof. The proof is a straightforward exercise. ■

We will now introduce some notions that will be useful to describe the time dependence during a typical shrinking of a subcritical droplet.

Let $\tau_0, \tau_1,..., \tau_n,...$ be random times in which our process σ_t visits (after a change) the set $\mathscr{Q}$ of configurations containing a unique octagon:

$$\tau_0 = \inf\{t \geq 0: \sigma_t \in \mathscr{Q}\}$$

$$\tau_{n+1} = \inf\{t > \tau_n: \sigma_t \in \mathscr{Q}\}, \qquad n = 0, 1, 2,...$$

Given l, $2 \leq l \leq l^*$, and $\varepsilon > 0$, we say that σ_t is an *ε-canonical contraction path* from $Q(l)$ to $Q(l-1)$ if

$$\tau_0 = 0, \qquad \sigma_0 \in Q(l), \qquad \sigma_{\tau_1} = Q^{(1)},..., \sigma_{\tau_{14}} = Q^{(14)}$$

where (i) $(\sigma_0, Q^{(1)},..., Q^{(14)} \equiv Q(l-1))$ is an element of the set $\mathscr{M}(l)$ of canonical contractions, and (ii) we have

$$\exp\{\beta[h(\hat{\mathscr{L}}^{(i)} - 1) - \varepsilon]\} < \tau_{i+1} - \tau_i < \exp\{\beta[h(\hat{\mathscr{L}}^{(i)} - 1) + \varepsilon]\}, i = 1,..., 14$$

where $\hat{\mathscr{L}}^{(i)} = \min_{j=i,...,8} \hat{\mathscr{L}}_j^{(i)}$ and $\hat{\mathscr{L}}_j^{(i)}$ are the lengths of the edges of $Q^{(i)}$.

Proposition 3C. Let G_1, G_2 be the set of octagons defined in Proposition 3A and let $\varepsilon > 0$. Then

$$P_{G_1 \cup G_2}(\tau_{-\underline{1}} > \exp\{\beta[h(l^* - 2) + \varepsilon]\} \xrightarrow[\beta \to \infty]{} 0 \qquad (3.127)$$

Moreover, with probability tending to one as $\beta \to \infty$:

(i) Starting from $G_1 \cup G_2$ our process will follow the remaining 13 steps of an ε-canonical contraction path up to $Q(l^* - 1)$.

(ii) Starting from $Q(l^* - 1)$ it will follow an ε-canonical contraction path up to $Q(l^* - 2)$ and so on up to $Q(2)$.

(iii) Finally, it will persist in $Q(2)$ for a time $t < \exp\{\beta(h + \varepsilon)\}$ and then, after an h-erosion, it will proceed downhill to $-\underline{1}$.

Proof. The validity of (3.127) is a corollary of Proposition 3A.

The statements (i) and (ii) follow from Lemma 3.5 and the fact that in a time less than $\exp\{\beta[h(l^* - 2) + \varepsilon]\}$, for ε sufficiently small, the only elementary events are, with high probability for β large, h-erosions and recoveries.

Statement (iii) is immediate. ∎

Remark. By Lemma 3.5 we get a lot of additional geometrical information about an ε-canonical contraction path that actually could have been included in the statement of Proposition 3C.

Finally, we consider the much simpler case of growth of supercritical octagons.

Proposition 4. Let $Q(D_1, D_2)$ be a standard octagon with $D_2 \leqslant D_1$ such that

$$L_2 = D_2 - 2(l^* - 1) \geqslant L^*$$

Let

$$\hat{\tau} = \tau_{Q(D_1, D_2+1) \cup Q(D_1+1, D_2)} \tag{3.128}$$

Then, for every $\varepsilon > 0$, one has

$$\sup_{\sigma \in Q(D_1, D_2)} P_\sigma(\hat{\tau} < \exp\{\beta(2J - 4K - h + \varepsilon)\} \text{ and } \hat{\tau} = \bar{\tau}) \xrightarrow[\beta \to \infty]{} 1 \tag{3.129}$$

where $\bar{\tau}$ has been defined in (3.69).

Proof. According to Lemma 3.3, we have

$$\begin{aligned} \inf_{\omega: \sigma_0 \to \mathscr{S}^c(D_1, D_2)} \sup_{\sigma \in \omega} H(\sigma) &= \min_{\sigma \in \partial\mathscr{S}(D_1, D_2)} H(\sigma) = E(D_1, D_2) \\ &= H(Q) + 2J - 4K - h \end{aligned} \tag{3.130}$$

In the present case, $L_2 > L^*$, the configurations in $\mathscr{S}(D_1, D_2)$ minimizing H on $\partial\mathscr{D}(D_1, D_2)$ are obtained by adding a unit-square protuberance to one of the coordinate edges of an octagon in $Q(D_1, D_2)$. Consider the time $t(\delta) = \exp\{\beta(2J - 4K - h + \delta)\}$. It follows from (3.130) and Lemma 3.4 that if δ is sufficiently small, any $\sigma \in \partial\mathscr{D} \backslash \mathscr{S}$ cannot be reached before $t(\delta)$ with a probability approaching one as $\beta \to \infty$. Let us use $\tilde{Q}_0$ to denote the octagon in $Q(D_1, D_2)$ corresponding to our initial condition, and use $\hat{S}_1$ to denote the saddle in $\mathscr{S}(D_1, D_2)$ obtained by adding to $\tilde{Q}_0$ the first unit-square protuberance to its vertical right side.

For every $\sigma \in \mathscr{D}(D_1, D_2)$, let $\mathscr{E}^0_{\sigma, t_0}$ and $\bar{\mathscr{E}}^0_{\sigma, \bar{t}_0}$ be defined as in (3.93) and (3.44) with $\bar{t}_0 = \exp\{\beta(2K - h + \bar{\delta})\}$ with $\bar{\delta}$ sufficiently small. We notice that, by Lemma 3.4, before the time $t(\delta)$ one cannot see, with probability approaching one as $\beta \to \infty$, any J-protuberance occurring on a configuration $\bar{\sigma} \in \mathscr{D}(D_1, D_2)$ that would differ from $\tilde{Q}_0$. Otherwise we would touch the configuration

$$\sigma' = \bar{\sigma} + J\text{-protuberance}$$

with $H(\sigma') - H(\tilde{Q}_0) \geqslant 2J - 4K - h + \Delta$ for some positive Δ. Thus we can apply the same argument as that used in the case $L_2 < L^*$. In this way it is easy to get (3.119).

Now, consider the setup for introducing our auxiliary Markov chains with $N=1$, $Q_1=\tilde{Q}_0$, $B_1=\mathscr{B}(\tilde{Q}_0)$, $S_2=\hat{S}_1$, and S_1 the saddle obtained from $\tilde{Q}_0$ by adding a K-protuberance to one of its oblique edges,

$$\min_{\sigma \in \partial \mathscr{B}(\tilde{Q}_0)} H(\sigma) = H(S_1)$$

For every $\sigma \in \mathscr{D}(D_1, D_2)$ consider the event (of growth)

$$\mathscr{E}_\sigma^g = (\bar{\mathscr{E}}^0_{\sigma, t_0}; \bar{\mathscr{F}}^{1,u}) \cap \mathscr{G}$$

where

$$\mathscr{F}^{1,u} = \{\xi_0^1 = \tilde{Q}_0; \xi_{v_1}^1 = \hat{S}_1\}$$
$$\mathscr{G} = \{\tilde{\tau}_{\partial B(\tilde{Q}_0)} \leqslant \bar{t}_u\}$$

and

$$\bar{t}_u = \exp\{\beta(2K - h + \tilde{\delta})\}$$

As in the case $L_2 < L^*$, we deduce the analog of (3.97). Namely, for every $\varepsilon > 0$ and β sufficiently large,

$$\inf_{\sigma \in \mathscr{S}(D_1, D_2)} P(\mathscr{E}_\sigma^g) \geqslant \alpha \equiv \exp\{-\beta[(2J - 4K - h) - (2K - h) + \varepsilon]\} \tag{3.131}$$

Hence, by a recurrence argument similar to the one given in estimates (3.90), (3.91), with the help of the strong Markov property, we get

$$P_\sigma(\tau_{\partial \mathscr{S}} > \exp\{\beta(2J - 4K - h + \varepsilon)\}) \xrightarrow[\beta \to \infty]{} 0 \tag{3.132}$$

for every $\varepsilon > 0$ and any $\sigma \in \mathscr{D}(D_1, D_2)$. Then, again with the help of the strong Markov property, we get the desired results in the same way as in the proof of Proposition 1 since, again,

$$P_{\bar{\sigma}}(\hat{\tau} = 1) > \frac{1}{|A|}, \quad P_{\bar{\sigma}}(\tau_{\mathscr{S}(D_1, D_2)} < \tau_{\mathscr{S}^c(D_1, D_2) \setminus \{\bar{\sigma}\}} \mid \hat{\tau} > 1) \xrightarrow[\beta \to \infty]{} 1 \tag{3.133}$$

Once more, we leave the details to the reader. ■

4. GLOBAL SADDLE POINT

Similarly as in ref. 8, the proof of our theorems about "escape time and optimal route" are based on the existence of a set $\mathscr{A}$ with the following properties:

(i) For every $\sigma \in \mathscr{A}$ and any $\varepsilon > 0$ one has

$$\lim_{\beta \to \infty} P_\sigma(\tau_{-\underline{1}} < \tau_{+\underline{1}}) = 1$$

and

$$\lim_{\beta \to \infty} P_\sigma(\tau_{-\underline{1}} < \exp\{\beta[h(D^* - 1) + \varepsilon]\}) = 1$$

(ii) Every path ω starting in $-\underline{1}$ and ending in $+\underline{1}$ has to pass through the boundary $\partial\mathscr{A}$ of $\mathscr{A}$ defined by

$$\partial\mathscr{A} = \{\sigma \notin \mathscr{A}, \text{ there exists } x \text{ such that } \sigma^{(x)} \in \mathscr{A}\}$$

(iii) The minimal energy in $\partial\mathscr{A}$ is attained for "protocritical" (global saddle) configurations $\sigma \in \mathscr{P}$—a single unit square attached to the longer coordinate side of a standard octagon $Q(D^* - 1, D^*)$—with the energy $E^* \equiv H(Q(D^* - 1, D^*)) + 2J - 4K - h$. All configurations in $\partial\mathscr{A}$ that are not of this form have the energy at least h higher.

As a first step toward the construction of the set $\mathscr{A}$ we construct for every configuration σ (from a certain class) a configuration $\bar{\sigma}$ such that

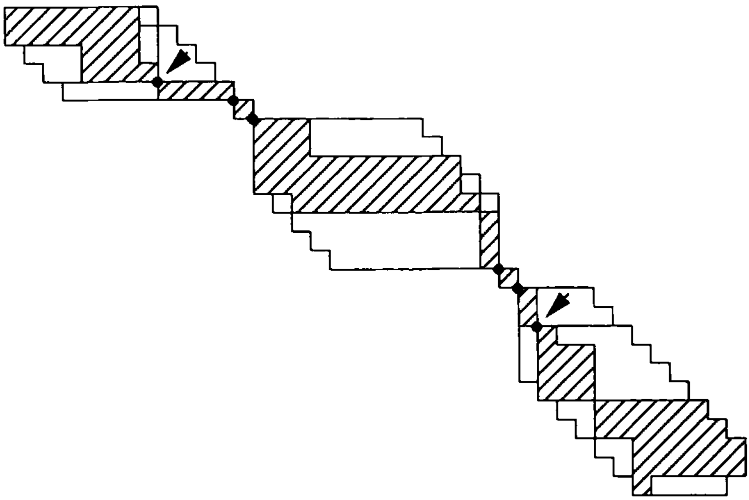
$$S\sigma \prec \bar{\sigma}$$

for every standard S. Here we use the natural order on the set of configurations:

$$\sigma_1 \prec \sigma_2 \quad \text{iff} \quad \{x: \sigma_1(x) = +1\} \subset \{x: \sigma_2(x) = +1\}$$

In other words, the configuration $\bar{\sigma}$, to be called *the blown up envelope of* σ, is the "maximal" one (and actually even larger) to which we can arrive by applying a standard sequence.

We begin by constructing $\bar{\sigma}$ in the case when σ corresponds to a single droplet $C = C(\sigma)$ [i.e., the set $C(\sigma)$ is connected]. Suppose also that the rectangular envelope $R(C)$ does not wind around the torus and consider the monotone envelope $M = M(C)$. The set M consists of monotonic blocks connected in corners, of the form shown on Scheme 2.1, where four edges meet. We say that such point is a *bottleneck* of the set M, once the intersection of the boundary ∂M with each of the two outside right angles touching in this point has at least along one side the length one (Scheme 4.1, where a thick dot indicates a bottleneck of set M; the points denoted by an arrow cease to be bottlenecks after "enveloping the components").



Scheme 4.1.

Disconnecting now the set M in bottlenecks, consider the union $M^{(1)}$ of octagonal envelopes of corresponding components. Some of the resulting points that were bottlenecks for M may not be such any more for the resulting set $M^{(1)}$. Considering only its bottlenecks, we repeat the procedure and iterate until the set of bottlenecks does not change. We use $N(C)$ to denote the droplet constructed above from the set C and call it a *string* (i.e., a string is a monotonic droplet whose components, after disconnecting its bottlenecks, are octagons with oblique sides $l_i \geqslant 1$—not necessarily larger than or equal to 2).

The configuration $\bar{\sigma}$ corresponding to the droplet $N(C(\sigma))$ is the sought blown up envelope with the desired properties.

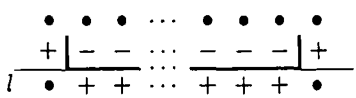
Lemma 4.1A. Let σ be a configuration with a single droplet $C(\sigma)$ contained in a rectangle that is not wrapped around the torus. Then (i) $H(\bar{\sigma}) \leqslant H(\sigma)$, (ii) $\sigma \prec \bar{\sigma}$ and $S\sigma \prec \bar{\sigma}$ for every standard sequence.

Proof. Since taking octagonal envelopes of components decreases the energy, for our proof of (i) it is sufficient to show that

$$H(\sigma) \geqslant H(M)$$

(we identify the droplet M with the corresponding configuration). We will show that there exists a sequence of configurations with decreasing energy and leading from σ to M.

First, we show that the holes inside $=C(\sigma)$ (minus spins inside $\partial_{\text{out}} C$; see Scheme 3.2) can be filled up. Consider the boundary of the holes, $\partial C \backslash \partial_{\text{out}} C$, and consider the lowermost horizontal line touching it. It touches it along a certain number of segments, with the spins around each of them necessarily taking the values shown in Scheme 4.2.



Scheme 4.2

Notice, in particular, the + spins below the line. If any of them were replaced by $-$, the line l would be pushed lower. The spins denoted by dots are arbitrary. Labeling the spins in the first row as σ_{I}, $\sigma_1,..., \sigma_k$, σ_{II} and those in the third line that are not fixed as σ_{III}, σ_{IV}, we get an energy decrease after flipping simultaneously all minus spins in the second row. Namely,

$$-\Delta H = 2\tilde{J} + \tilde{J} \sum_{i=1}^{k} (1+\sigma_i) + K(1+\sigma_{\text{I}}+\sigma_{\text{III}}+\sigma_2) + K(1+\sigma_{\text{II}}+\sigma_{\text{IV}}+\sigma_{k-1}) + K \sum_{i=2}^{k-1} (2+\sigma_{i-1}+\sigma_{i+1}) + hk$$

whenever $k \geqslant 2$ and

$$-\Delta H = 2\tilde{J} + \tilde{J}(1+\sigma_1) + K(\sigma_{\text{I}}+\sigma_{\text{II}}+\sigma_{\text{III}}+\sigma_{\text{IV}}) + h$$

for $k=1$. In both cases its minimum is attained if all spins denoted by dots are minus, $\sigma_i = -1$, $i=1,..., k$, $\sigma_{\text{I}} = \sigma_{\text{II}} = \sigma_{\text{III}} = \sigma_{\text{IV}} = -1$, and we get

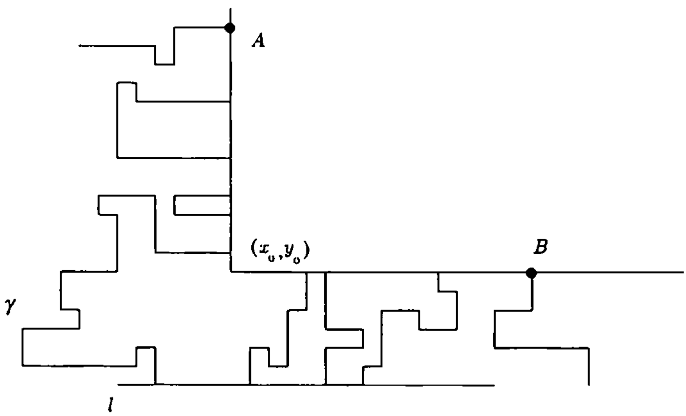
$$-\Delta H \geqslant 2\tilde{J} - 4K + h > 0$$

according to our assumption $K \leqslant \frac{1}{2}(\tilde{J}-h)$ (cf. Lemma 3.1). Flipping thus all considered minus spins above the line l, it will be pushed higher. Iterating the process, we finally erase all holes in C, decreasing at the same time the energy of the configuration. Hence, we can suppose that $\partial_{\text{out}} C = \partial C$.

In a similar way we can prove that filling the droplet up to M, we further decrease the energy. Namely, consider a right angle, say $\{(x, y) \in \mathbb{R}^2;\ \mu x \geqslant x_0,\ y \geqslant y_0\}$, such that C does not intersect its interior and such that it touches ∂C in at least two distinct points. Consider those two such points

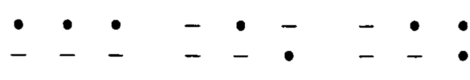
A and B whose distance is maximal (see Scheme 4.3—even though we show here A and B belonging one to the horizontal and one to the vertical side of the angle, nothing prevents them from belonging on the same side).

Considering now an order on the path winding around ∂C (as in the proof of Lemma 2.1), take the portion γ between A and B and the lowermost horizontal line l touching γ. If l does not contain the point (x_0, y_0), we can decrease the energy by flipping intervals of minuses just above the line in the same way as we did when erasing the holes of C. Iterating the procedure unless l passes through (x_0, y_0) and repeating then all with the leftmost vertical line, we finally replace γ by a corresponding segment of the boundary of the angle. Repeating the same for all relevant angles, we finally obtain the configuration M and prove thus (i).



Scheme 4.3

To prove (ii) we assume the contrary and consider a standard sequence S for which $S\sigma \prec \bar{\sigma}$ does not hold. Take the first site $x \in S$ outside $C(\bar{\sigma})$, $x \notin C(\bar{\sigma})$. Considering an arbitrary configuration ξ inside $C(\bar{\sigma})$ and $\xi(y) = -1$ for all $y \notin C(\bar{\sigma})$, we will show that the spin flip $\xi(x) = -1 \to +1$ always increases energy. This is clear if x is not attached to the boundary of $C(\bar{\sigma})$. Taking into account that $C(\bar{\sigma})$ is a string, the local situation around x attached to $\partial C(\bar{\sigma})$ is among the following ones (with the site x in the center of the square and up to rotations and reflections):



A) B) C)

Notice that the case A covers, for example, the situations

$$\begin{matrix} \bullet & - & - \\ - & - & - \\ - & - & - \end{matrix}\quad , \quad \begin{matrix} - & \bullet & - \\ - & - & - \\ - & - & - \end{matrix}$$

while B, e.g., covers

$$\begin{matrix} - & \bullet & - \\ - & - & \bullet \\ - & - & - \end{matrix}\quad , \quad \begin{matrix} - & \bullet & - \\ - & - & - \\ - & - & \bullet \end{matrix}$$

A glance at the catalogue of stable situations in Section 3 assures us that in all three cases the minus spin in the center is stable irrespective of the values of spins at sites denoted by dots, in contradiction with the assumption that the spin flip $\xi(x) = -1 \rightarrow +1$ decreases the energy. The stability in case A follows from (a) of the catalogue, B from (a), (e), and (g), and C from (a), (d), and (g). ■

Notice that the string $\bar{\sigma}$ might still contain unstable lattice sites—namely thos plus spins that are surrounded by four nearest neighbor minuses. When applying a standard sequence S, one will eventually erase them, obtaining a union of octagons contained in $C(\bar{\sigma})$.

Next we proceed to a case of a configuration σ consisting of several components. To treat this case we shall repeatedly use the following lemma to evaluate the sum of energies of two configurations by the energy of the configuration whose area of plusses is the union of areas of plusses of the two concerned configurations. For configurations σ_1 and σ_2 we define the minimum $\sigma_1 \wedge \sigma_2$ and the maximum $\sigma_1 \vee \sigma_2$ with respect to the order $\prec$. Namely, these configurations are given by taking the minimum and maximum, respectively, of σ_1 and σ_2 site by site. Then we have the standard inequality (a base of the FKG inequality):

Lemma 4.2. For any σ_1, σ_2 and the Hamiltonian (2.2) one has

$$H(\sigma_1) + H(\sigma_2) \geqslant H(\sigma_1 \vee \sigma_2) + H(\sigma_1 \wedge \sigma_2) \tag{4.1}$$

Proof. Using the equality

$$a + b = \max(a, b) + \min(a, b) \tag{4.2}$$

we get

$$h \sum \sigma_1(x) + h \sum \sigma_2(x) = h \sum (\sigma_1 \vee \sigma_2)(x) + h \sum (\sigma_1 \wedge \sigma_2)(x)$$

The remaining two sums in the definition (2.2) of the Hamiltonian consist of terms of the form $\sigma(x)\,\sigma(y)$ and the inequality (4.1) will be verified once we show that

$$\sigma_1(x)\,\sigma_1(y)+\sigma_2(x)\,\sigma_2(y)\leqslant(\sigma_1\vee\sigma_2)(x)(\sigma_1\vee\sigma_2)(y)$$
$$+(\sigma_1\wedge\sigma_2)(x)(\sigma_1\wedge\sigma_2)(y) \qquad (4.3)$$

If $\sigma_1(x)=\sigma_2(x)$ we have an equality in (4.3) by (4.2). If $\sigma_1(x)\neq\sigma_2(x)$, say $\sigma_1(x)=+1$, $\sigma_2(x)=-1$, the inequality (4.3) follows from

$$\sigma_1(y)-\sigma_2(y)\leqslant\max(\sigma_1(y),\sigma_2(y))-\min(\sigma_1(y),\sigma_2(y)) \quad \blacksquare$$

Consider now a configuration σ whose set $C(\sigma)$ has several components $C_1,\ldots,C_k$. Let us suppose that the rectangles $R(C_1),\ldots,R(C_k)$ are not wrapped around the torus and let $\bar\sigma_1,\ldots,\bar\sigma_k$ be the strings obtained as blown up envelopes of the configurations $\sigma_1,\ldots,\sigma_k$ corresponding to droplets $C_1,\ldots,C_k$. Using $\bar H$ to denote the relative energy with respect to the energy of $-\underline{1}$, i.e., $\bar H(\sigma)=H(\sigma)-H(-\underline{1})$, by Lemma 4.1A we have

$$\bar H(\sigma)\geqslant\bar H(\bar\sigma_1)+\cdots+\bar H(\bar\sigma_k) \qquad (4.4)$$

If the droplets $N_1,\ldots,N_k$ corresponding to configurations $\bar\sigma_1,\ldots,\bar\sigma_k$ were isolated, we would define $\bar\sigma$ by taking $C(\bar\sigma)=\bigcup_{i=1}^k N_i$.

If the droplets $N_1,\ldots,N_k$ are not isolated, we first glue them together and repeat the procedure. Namely, if two droplets N, N' have a nonempty intersection, consider the droplet $N\cup N'$. By Lemma 4.2 we have

$$H(N)+H(N')\geqslant H(N\cup N')+H(N\cap N')$$

Let us suppose that the rectangular envelopes $R(N)$ and $R(N')$ are subcritical (i.e., not winding around the torus and with minimal sides not exceeding D^*). Then $\bar H(N\cap N')\geqslant 0$. Indeed, applying any standard sequence to $N\cap N'$, we get a union of subcritical octagons of even lower energy. And the energy of a subcritical octagon with respect to the energy of the configuration $-\underline{1}$ is positive. Thus

$$\bar H(N)+\bar H(N')\geqslant\bar H(N\cup N') \qquad (4.5)$$

Using this observation, we can consider the components $\tilde N_1,\ldots,\tilde N_k$ of $N_1\cup N_2\cup\cdots\cup N_k$ and show that

$$\bar H(\sigma)\geqslant\bar H(\tilde N_1)+\cdots+\bar H(\tilde N_l) \qquad (4.6)$$

once we suppose that the circumscribed rectangles of $\tilde N_1,\ldots,\tilde N_l$ are subcritical.

Further, we say that two droplets $\tilde{N}$ and $\tilde{N}'$ *stick* together if there exists a site $x \notin \tilde{N} \cup \tilde{N}'$ (necessarily touching $\partial\tilde{N}$ as well as $\partial\tilde{N}'$) and a configuration[6] inside $\tilde{N}$ and $\tilde{N}'$ such that, flipping $-1 \to +1$ in x, we decrease the energy of the configuration corresponding to $\tilde{N} \cup \tilde{N}'$ and obtain a connected set $\tilde{N} \cup \tilde{N}' \cup q(x)$ [here $q(x)$ is the unit square around x]. To avoid ambiguities, we choose the pairs of droplets sticking together as well as the particular site x to be flipped using some canonical order—say, lexicographic order of the uppermost left corner of the droplet and the same order for the site x. Flipping the concerned site x and iterating the procedure, we obtain a set $C'_1,\dots, C'_m$ of disjoint droplets such that neither pair of them sticks together. Clearly,

$$C'_1 \cup \cdots \cup C'_m \supset N_1 \cup \cdots \cup N_k$$

and

$$H(\sigma) \geqslant H(C'_1) + \cdots + H(C'_m)$$

Supposing again that the circumscribed rectangles $R(C'_1),\dots, R(C'_m)$ are subcritical (and in particular are not winding around the torus), we repeat the same procedure as when we started from σ. Iterating it, we finally get a set of disjoint $A_1,\dots, A_n$ such that no pair of them sticks together. If their circumscribed rectangles are not wrapped around the torus and are subcritical, we say that the original configuration σ is *acceptable* and define $\bar{\sigma}$ so that $C(\bar{\sigma}) = A_1 \cup \cdots \cup A_n$. Clearly,

$$H(\sigma) \geqslant H(\bar{\sigma})$$

Thus we get the following definitive result.

Lemma 4.1B. Let σ be an acceptable configuration. Then:

(i) $H(\bar{\sigma}) \leqslant H(\sigma)$.

(ii) $S\sigma \prec \bar{\sigma}$ for every standard sequence.

(iii) $\sigma \prec \bar{\sigma}$ and the summary length of boundaries of all droplets of σ inside a component A_i of $C(\bar{\sigma})$ is at least as large as the length of the boundary of the circumscribed rectangle $R(A_i)$.

[6] It is easy to see that if such a configuration exists, one can take the configuration with plusses in $\tilde{N} \cup \tilde{N}'$, but we do not need this fact.

Proof. The statement (i) was already proven during the construction.

To prove (ii) we use the same argument as when proving (ii) in Lemma 4.1A. One has only to observe that, considering a site x attached to a component A_i of $C(\bar{\sigma})$ and supposing that another component A_j is touching the unit square $q(x)$, the spin flip $\sigma(x) = -1 \to +1$ necessarily increases energy irrespective of the configuration inside A_i and A_j. Indeed, if a configuration inside $A_i \cup A_j$ existed so that the energy decreases, the strings A_i and A_j would stick together, which is not the case.

To prove (iii) we first notice that in the first stages of the construction of $\bar{\sigma}$ we only decreased the number of bounds—the length of the boundary of $\cup \tilde{N}_i$ is not larger than the boundary of the original $C(\sigma)$. Further, whenever gluing two sticking components we do not change the length of the boundary. ■

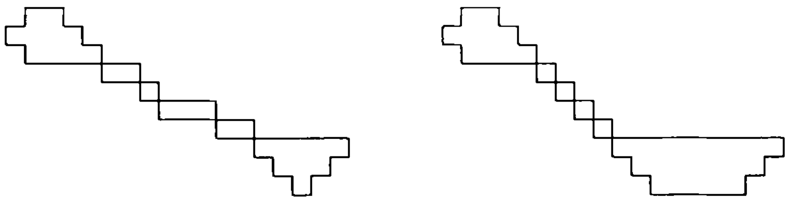
We say that a string A is *ephemere* if $\min(D_1(R(A)), D_2(R(A))) < 3l^* - 3$. Strings that are not ephemere energy larger than or equal to the energy of the standard octagon inscribed in their rectangular envelope.

Lemma 4.3. Let A be a string that is not ephemere and R be its rectangular envelope, $R = R(A)$, and Q be the standard octagon with $R(Q) = R$. Then

$$H(A) \geqslant H(Q) \tag{4.7}$$

Proof. The string A consists of certain number of octagonal blocks touching in bottlenecks. Actually, the inside blocks—those not touching the boundary of the rectangle $R(A)$—are in general "hexagons," while the two corner blocks—those touching the boundary of $R(A)$—are in general "heptagons" (cf. Scheme 4.1). The energy of a string does not change if we reorder the inside blocks. In the case when any corner block contains also the vertex of the rectangle $R(A)$ (it is a "hexagon"), it can be interchanged with any inside block without changing the energy of the configuration.

After a reordering some bottlenecks may disapear and one can decrease the energy of the corresponding configuration by replacing a group of blocks attached at points that are no longer bottlenecks by an octagonal envelope. Using this observation in an iterative manner, one can finally replace the string A by a string $\bar{A}$ such that $R(A) = R(\bar{A})$, $H(A) \geqslant H(\bar{A})$, and all the inside blocks of $\bar{A}$ are unit squares. Indeed, starting from an arbitrary string A, one easily gets a string whose all inside blocks are rectangles such that, say, the vertical side of all of them is of length one. We decrease further the energy and get $\bar{A}$ by shrinking all inside blocks to unit squares and expanding in the corresponding manner one of the corner blocks (see Scheme 4.4). This last step does not change the number of corners while increasing the area of the droplet.



Scheme 4.4

To evaluate now the energy of such a droplet $\bar{A}$ with d inside unit square blocks, we will consider two cases: (i) $d \geqslant \frac{1}{2}(3+\sqrt{5})\, l^* + 3$, or (ii) $d < 3l^* - 2$. These two cases cover all values of d once $(3-\sqrt{5})\, l^* > 10$, i.e., if $K > 7h$.

In the first case, the energy decreases if we replace all inside (unit square) blocks by a single hexagon with two oblique sides of length l^*. Indeed, to prove this we have to show that [cf. (3.18) and (3.6)]

$$-hd - (d-1)\, 6K - 2K > -hd^2 + 2F(l^*)$$

or equivalently, using (3.6) and (2.18),

$$P(d) \equiv d^2 - d(3l^* - 2 + 3\eta) + 2(l^* - 1 + \eta) + [(l^* - 1)^2 + \eta(2l^* - 1)] > 0$$

Taking into account that $\eta \in (0, 1)$, the discriminant of the quadratic expression $P(d)$ is bounded from above by $5(l^* + 2)^2$. Observing further that the term d^2 in $P(d)$ has a positive coefficient, the sought inequality is fulfilled once d exceeds the larger one from the solutions of the equation $P(d) = 0$. This solution is bounded from above by

$$\frac{1}{2}\,[3l^* + 1 + \sqrt{5}(l^* + 2)] \leqslant \frac{3+\sqrt{5}}{2}\, l^* + 3$$

The resulting string contains a hexagon that is attached to (if any) two corner blocks in points that are not bottlenecks and thus the energy decreases if we replace this string by its octagonal envelope. Its energy can be further lowered by replacing it by the standard octagon Q with the same circumscribed rectangle $R(A)$ (cf. Lemma 3.2).

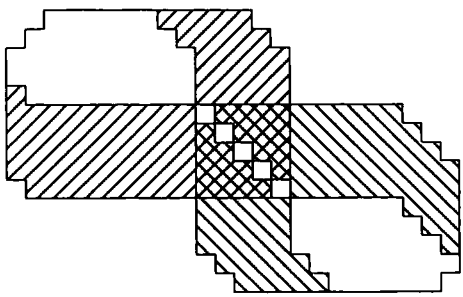
On the other hand, if $d < 3l^* - 2$, we consider the monotonic set $\bar{M}$ constructed from $\bar{A}$ as the union of the $d \times d$ square $\bar{Q}$ circumscribed to the union of inside (unit square) blocks with the corner blocks and the union of all their shifts, up to the distance d, along those sides of the rectangle $R(A)$ they are touching (see Scheme 4.5).

Using D to denote $\min(D_1(R(A)), D_2(R(A)))$, the area covered by $\bar{M}\backslash\bar{A}$ is at least

$$2Dd - d^2 + d$$

On the other hand, the droplet $\bar{M}$ has by $6(d-1)$ less corners and the replacement of $\bar{A}$ by $\bar{M}$ is favorable once

$$[(2D+1)\,d - d^2]\,h > 6(d-1)\,K$$



Scheme 4.5

This inequality is fulfilled if

$$(2D + 1 - d)\,h > 6K = 3(l^* - 1 + \eta)\,h$$

Thus, it is enough to take d such that

$$d < 2D + 4 - 3l^*$$

Since $D \geqslant 3l^* - 3$, the inequality is fulfilled once $d < 3l^* - 2$. The energy of the droplet $\bar{M}$ is lower than that of its octagonal envelope and this in turn is lower than the energy of the standard octagon Q. ■

Now we are ready to define the set $\mathscr{A}$.

Consider an acceptable configuration σ and the corresponding $\bar{\sigma}$ with nonsticking components $A_1,..., A_k$. Consider further the family of sets $R_1,..., R_k$, where $R_i = A_i$ if A_i is ephemere; otherwise R_i is the rectangular envelope of A_i. Two strings A_i, A_j are said to *interact* if at least one of them is not ephemere and the sets R_i and R_j (but not necessarily A_i and A_j) stick together (or intersect).[7]

A family of strings $A_1,..., A_k$ is said to form a *chain* $\mathscr{C}$ if every pair (A_i, A_j) of them can be linked by a sequence $\{A_{i_1},..., A_{i_n}\}$ of pairwise inter-

[7] However, if A_i and A_j stick together, they necessarily interact.

acting strings from $\mathscr{C}$; $A_{i_1}=A_i$, $A_{i_n}=A_j$, and A_{i_l} and $A_{i_{l+1}}$ are interacting for all $l=1,\dots,n-1$.

Given a collection of chains $\mathscr{C}_1,\dots,\mathscr{C}_n$ corresponding to the family $A_1,\dots,A_k$, we start the following iterative procedure:

1. The chains $\mathscr{C}_j^{(1)}$ of the "first generation" are identical to $\mathscr{C}_j$, $j=1,\dots,n$.
2. Having defined $\mathscr{C}_j^{(r)}$, we construct the sets $R_j^{(r)}$ as the rectangular envelopes of the unions

$$\bigcup_{R\in\mathscr{C}_j^{(r)}} R$$

whenever $\mathscr{C}_j^{(r)}$ contains at least two sets; we put $R_j^{(r)}=R$ if $\mathscr{C}_j^{(r)}=\{R\}$.

We define $\mathscr{C}_j^{(r+1)}$ as the maximal chains of the family of sets $\{R_j^{(r)}\}$.

Iterating this procedure, we reach a family of chains, each consisting of a single set. We call them *complete sets of the configuration* σ. It is easy to observe that every complete set from the resulting family $\bar{R}_1,\dots,\bar{R}_s$ is either a rectangle containing a certain amount of original strings $A_1,\dots,A_k$ or it is one of ephemere strings contained in the family $A_1,\dots,A_k$.

We introduce $\mathscr{A}$ as the set of all those acceptable configurations for which all complete sets $\bar{R}_1,\dots,\bar{R}_s$ are subcritical—they can be placed in a rectangle that is not wrapped around the torus and whose minimal side does not exceed D^*-1.

In the remaining part of the present section we shall verify the properties (i)–(iii) of the set $\mathscr{A}$. It is easy to see that the property (ii) of $\mathscr{A}$ is obvious from the definition, while the property (i) follows from Propositions 1–3.

To prove the property (iii) we first consider the configuration $\hat{\sigma}$ obtained by placing plus spins at all lattice sites in $\bar{R}_i$ for ephemere $\bar{R}_i=\bar{A}_i$ and in the standard octagon Q_i inscribed in $\bar{R}_i$ for $\bar{R}_i$ that is not ephemere. We will repeatedly use the bound

$$H(\sigma)\geqslant H(\hat{\sigma}) \tag{4.8}$$

This follows, iterating with the help of Lemma 4.3, from the following lemma.

Lemma 4.4. Let A' and A'' be a pair of interacting strings and let ξ be the configuration with the set of plusses coinciding with $A'\cup A''$. Suppose that the rectangular envelope R of the set $A'\cup A''$ is subcritical. Then

$$H(\xi)\geqslant H(\hat{\xi}) \tag{4.9}$$

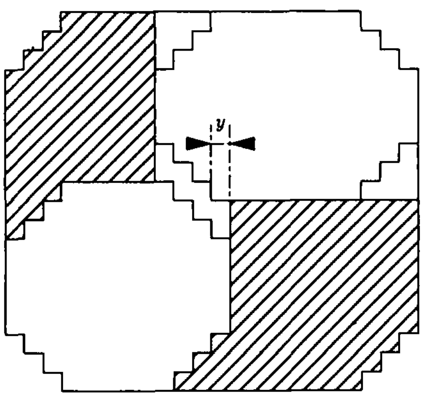
where ξ is the configuration with plusses in the standard octagon Q inscribed into R.

Proof. Suppose first that both A' and A'' are not ephemere and consider the standard octagons Q' and Q'' inscribed into the rectangular envelopes R' and R'' of A' and A'', respectively. Clearly, by Lemma 4.3, $\bar{H}(\xi) \geq \bar{H}(Q') + \bar{H}(Q'') \geq \bar{H}(Q' \cup Q'')$. If Q' and Q'' stick together (i.e., Q' and Q'' intersect each other or there exists a site x such that flipping the spin at x decreases the energy), the energy of the single droplet $Q' \cup Q'' \cup q(x)$ (resp. $Q' \cup Q''$ if Q' and Q'' intersect) is smaller than $H(\xi)$. It can be further lowered by taking its monotonic envelope and, finally, by replacing it with the octagon Q. If Q' and Q'' do not stick together, there does not exist a site x such that the unit square $q(x)$ intersects the boundaries of both Q' and Q'' along a bond and we can see that the area of Q is by at least

$$\min_{y \in [0, l^* - 1]} [(2(l^* - 1) - y)(2(l^* - 1) + y) + (2(l^* - 1) + y)(2(l^* - 1) - y)]$$

$$\geq \min_{y \in [0, l^* - 1]} [8(l^* - 1)^2 - 2y^2] \geq 6(l^* - 1)^2$$

larger than the area of $Q' \cup Q''$ (see Scheme 4.6).



Scheme 4.6

Taking into account that $4F(l^*) \geq -2(l^* - 1)\,h$, we get

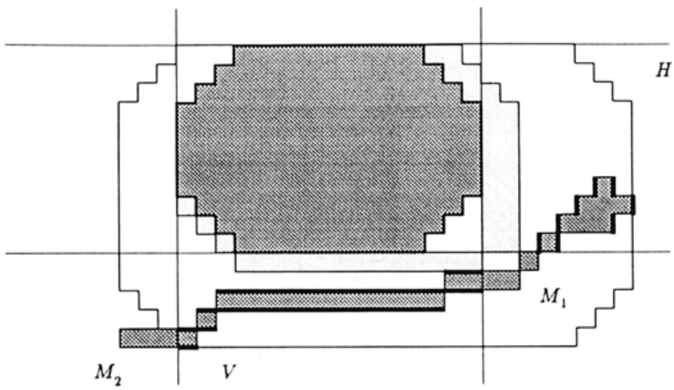
$$6(l^* - 1)\,h > -4F(l^*)$$

and thus the energy decreases once we replace the configuration ξ by Q. (We are not even using the fact that the overall number of bonds in the boundaries of Q' and Q'' might be larger than the number of bonds in the boundary of Q leading to an even larger decrease of energy.)

It remains to consider the case when one of the strings, say A'', is ephemere. (See Scheme 4.7.) The only nontrivial situation is when Q' and A'' do not stick together. When replacing the configuration ξ (with energy not lower than that of the union of Q' and A'') by Q, we have to compensate the loss of all corners in $\partial A''$ by the surplus area $|Q| - |Q' \cup A''|$ and surplus length of the boundary $|\partial Q'| + |\partial A''| - |\partial Q|$. Consider now the strips H and V obtained as the union of all horizontal and vertical shifts of Q', respectively. It is easy to see that the surplus length is at least the number N_h of horizontal bonds of $\partial A''$ in V plus the number N_v of vertical bonds of $\partial A''$ in H. Each corner of $\partial A''$ in V is linked with at least one horizontal bond of $\partial A''$ in V. The number of corners attached to N_h horizontal bonds in V is at most $6(N_h/2) + 6$. Their loss is compensated by the surplus of those N_h bonds since

$$(3N_h + 6)\, K \leqslant 6N_h K \leqslant N_h J$$

Similarly for the vertical bonds in H.



Scheme 4.7

Thus it remains to compensate for corners not contained in the closed set $H \cup V$. Consider the four quadrants—components of $(H \cup V)^c$. The set A'' intersects at most two of them. We use U_1, U_2 to denote these two quadrants and $M_i = U_i \cap A''$, $i = 1, 2$. Moreover, there is at most one quadrant of $(H \cup V)^c$, say U_1, both of whose sides intersect A''. Without loss of generality we can suppose that M_2 does not intersect H and the circumscribed rectangle $R(M_2)$ to M_2 is $m_2 \times n_2$. Then there is at least $\min(m_2, n_2)(3l^* - 2)$ of surplus area of $|Q| - |Q' \cup A'|$ in the component of $H \backslash Q'$ that touches U_2. On the other hand, at most $\min(m_2, n_2)\, l^*$ of the area may be lost from the portion of M_2 sticking out of Q. Using b_2 to denote the number of bottlenecks in M_2, there is also at least $b_2(3l^*-2)$ of surplus area of $|Q|-|Q'\cup A'|$ in the component of $V\backslash Q'$ that touches U_2. At the same time, the number of corners in M_2 (that are not at the boundary of U_2 and thus were not accounted for before) is at most $6b_2+4(\min(m_2, n_2)-1-b_2)$ and we have

$$6b_2 K+4(\min(m_2, n_2)-1-b_2)\,K \leqslant [(3l^*-2)\min(m_2, n_2)-\min(m_2, n_2)\,l^*+b_2(3l^*-2)]\,h$$

To analyze the quadrant U_1, we distinguish two possibilities. First, suppose that M_1 contains at least one bottleneck, consider the quadrant U', with vertex in this bottleneck, that contains the octagon Q', and use n_h and n_v to denote the number of horizontal and vertical bonds in M_1, respectively. The number of corners of M_1 is at most $3\min(n_h, n_v)$. At the same time the surplus of the area $|(Q\backslash(Q'\cup A''))\cap U'\cap(H\cup V)|$ is at least $\frac{1}{2}(n_h+n_v)(3l^*-2)$. To get the sought bound we just notice that

$$3\min(n_h, n_v)\,K=\frac{3}{2}\min(n_h, n_v)(l^*-1+\eta)\,h$$
$$<\min(n_h, n_v)\frac{3}{2}\,l^*h\leqslant\frac{n_h+n_v}{2}(3l^*-2)\,h$$

If there is no bottleneck of A'' in U_1, the portion of A'' between two bottlenecks containing the set M_1 is clearly contained in a half-plane P, the oblique boundary ∂P of which passes through the vertex of U_1 and is orthogonal to the axis of symmetry of U_1. The number of corners of M_1 not accounted for before is in this case at most $2\min(n_h, n_v)$. On the other hand, the area $|(Q\backslash(Q'\cup A''))\cap P^c\cap(H\cup V)|$ is at least $\frac{1}{2}\min(n_h, n_v)(3l^*-2)$ and

$$2\min(n_h, n_v)\,K\leqslant\frac{\min(n_h, n_v)}{2}(3l^*-2)\,h \quad \blacksquare$$

To verify the property (iii) of $\mathscr{A}$, let $\xi\in\partial\mathscr{A}$ and $\sigma\in\mathscr{A}$ be such that $\xi=\sigma^{(x)}\notin\mathscr{A}$. The mapping, assigning to a configuration ζ its complete sets, is monotonic in the ordering $\prec$ on configurations and the ordering by inclusion on complete sets. As a consequence, the value $\sigma(x)$ is necessarily -1; otherwise $\sigma\in\mathscr{A}$ would imply also $\xi\in\mathscr{A}$. Moreover, the site x lies outside of all complete sets $\bar{R}_1,\dots,\bar{R}_s$ of σ. Among the complete sets of ξ there exists a rectangle $\bar{R}(D_1, D_2)$ with the following properties: (i) $\bar{R}$ is supercritical, $\min(D_1, D_2) \geqslant D^*$, and (ii) it contains the site x and several complete sets $\bar{R}_i$, say $\bar{R}_1,\dots,\bar{R}_k$, of σ; the remaining complete sets $\bar{R}_{k+1},\dots,\bar{R}_s$ of σ are also complete sets of ξ.

Our aim now is to prove that

$$H(\xi) \geqslant H(Q(D^*, D^*)) + h(D^* - 1) - 4K = H(Q(D^* - 1, D^*)) + 2J - 4K - h \tag{4.10}$$

If $\bar{R}$ is winding around the torus, referring briefly to (4.8), the inequality (4.10) is clearly satisfied. Thus, let us suppose that $\bar{R}$ is not wrapped around the torus. Consider first the configuration $\tilde{\xi}$, $H(\xi) \geqslant H(\tilde{\xi})$, obtained by restricting the configuration ξ to the union $\bigcup_{i=1}^{k} \bar{R}_i \cup q(x)$, where $q(x)$ is the unit square centered at the site x [i.e., considering ξ to be plus on $\bigcup_{i=1}^{k} \bar{R}_i \cup q(x)$ and taking minuses outside; we will see in a moment—Eq. (4.11)—that the energy decreases when skipping the subcritical sets $\bar{R}_{k+1},\dots,\bar{R}_s$]. Further, consider the set $C^{(0)}$ consisting of the union of the unit square $q(x)$ and those sets among $\bar{R}_1,\dots,\bar{R}_k$ that have a common edge with $q(x)$. Let us take the rectangular envelope $\tilde{R}^{(1)}$ of $C^{(0)}$ if $C^{(0)}$ is not ephemere and $\tilde{R}^{(1)} = C^{(0)}$ for ephemere $C^{(0)}$, and distinguish two cases: either the set $\tilde{R}^{(1)}$ is supercritical or it is not. [Notice that both $C^{(0)}$ and $\tilde{R}^{(1)}$ may actually coincide with $q(x)$.]

If $\tilde{R}^{(1)}$ is supercritical, we decrease the energy of $\tilde{\xi}$ further by erasing all the remaining sets from $\bar{R}_1,\dots,\bar{R}_k$ that were not contributing to the set $C^{(0)}$ and considering the configuration $\xi^{(0)}$ yielded by the restriction of ξ to the set $C^{(0)}$. To see this, notice first that for any subcritical $\bar{R}_i$, the energy of the restriction ξ_i of ξ to $\bar{R}_i$, $\xi_i = \xi \restriction \bar{R}_i$, can be bounded from below as

$$\bar{H}(\xi_i) \geqslant 2K \tag{4.11}$$

Indeed, if $\bar{R}_i$ is subcritical and nonephemere, and denoting $\bar{D}_2 = D_2(\bar{R}_i) \leqslant D_1 = D_1(\bar{R}_i)$, we have

$$\begin{aligned}\bar{H}(\xi_i) &\geqslant 2J(\bar{D}_1 + \bar{D}_2) - h\bar{D}_1\bar{D}_2 - 2h(l^* - 1)^2 \\ &= 2J\bar{D}_1 - h\bar{D}_1\bar{D}_2 + 2J\bar{D}_2 - 2h(l^* - 1)^2 \\ &\geqslant h(l^* - 1)[(D^* - 1)\,3 - 2(l^* - 1)] \geqslant 2K\end{aligned} \tag{4.12}$$

If $\bar{R}_i$ is ephemere, we have

$$\bar{H}(\xi_i) \geqslant 2J(\bar{D}_1 + \bar{D}_2) - 6K\bar{D}_2 - h\bar{D}_1(3l^* - 3) + 2K \geqslant 2K \tag{4.13}$$

once $K \leqslant J/3$. Thus, the energy decreases if we erase the configuration in those $\bar{R}_i$ that are not contributing to $C^{(0)}$—the bound $2K$ on the right-hand side of (4.11) is needed in case $\bar{R}_i$ is touching $q(x)$ in a corner.

Observing that $\bar{R}_1, ..., \bar{R}_k$ are not interacting, there can be at most two sets, say $\bar{R}_1$, $\bar{R}_2$, contributing to $C^{(0)}$ [i.e., intersecting $q(x)$ along its edge].

Let us suppose first that $\bar{R}_1$ is the only set from $\bar{R}_1, ...,$ contributing to $C^{(0)}$. Recalling that $\bar{R}_1$ is subcritical and $\tilde{R}^{(1)}$ is supercritical at the same time, we infer that the set $\bar{R}_1$ must be nonephemere and, moreover, it is necessarily the rectangle $(D^* - 1) \times D$ [or $D \times (D^* - 1)$] with $D \geqslant D^*$ and with the unit square $q(x)$ attached to its longer side. Consider the standard octagon $\bar{Q}_1$ inscribed into $\bar{R}_1$. The configuration $\xi^{(0)}$ restricted to $\bar{R}_1$ is actually the configuration $\sigma^{(0)}$ obtained as the restriction of σ to $\bar{R}_1$ with the energy not lower than the energy of $\bar{Q}_1$ [cf. (4.8)], $H(\sigma^{(0)}) \geqslant H(\bar{Q}_1)$. If $q(x)$ does not touch $C(\sigma^{(0)})$, then

$$\bar{H}(\xi^{(0)}) = \bar{H}(q(x)) + \bar{H}(\sigma^{(0)}) \geqslant \bar{H}(q(x)) + \bar{H}(\bar{Q}_1) \tag{4.14}$$

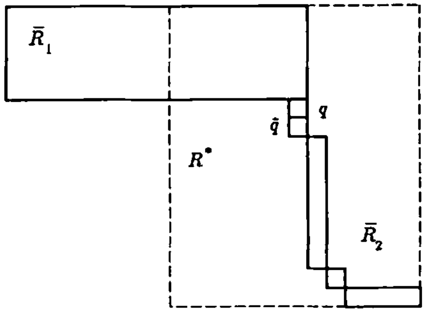
and the bound (4.10) follows since $\bar{H}(q(x)) = 4J - 4K - h > 2J - 4K - h$ and $H(\bar{Q}_1) \geqslant H(Q(D^*, D^* - 1))$. If $q(x)$ touches $C(\sigma^{(0)})$, then the monotone envelope of $\xi^{(0)}$ is a union of a monotone configuration M in $\bar{R}_1$ with the unit square $q(x)$ sharing a common edge with M. Hence

$$H(\xi^{(0)}) \geqslant H(M) + 2J - 4K - h \tag{4.15}$$

Using the bound $H(M) \geqslant H(\bar{Q}_1)$ according to Lemma 3.2, we get the sought inequality.

If there are two sets, $\bar{R}_1$ and $\bar{R}_2$, contributing to $C^{(0)}$, then at least one of them must be ephemere (otherwise $\bar{R}_1$ and $\bar{R}_2$ would interact). Suppose first that $\bar{R}_1$ is nonephemere and $\bar{R}_2$ is ephemere. Considering the half-plane p containing $\bar{R}_1$ whose boundary contains the edge separating q from $\bar{R}_1$, then, necessarily, $\bar{R}_2 \subset \mathbb{R}^2 \backslash p$. Moreover, the first row along the boundary of the half-plane p (the difference $p' \backslash p$, where p' is the half-plane p shifted by the unit vector orthogonal to its boundary) does not intersect the interior of $\bar{R}_2$ [it contains only the square $q(x)$], the square $q(x)$ is attached to $\bar{R}_1$ in a corner, and the second row contains a single square, to be denoted $\bar{q}$, of $\bar{R}_2$ (cf. Scheme 4.8).

Since $\tilde{R}^{(1)}$ is supercritical, there exists a $D \times D^*$ (or $D^* \times D$) rectangle R^* with $D \geqslant D^*$ such that (i) R^* contains $q \cup \bar{R}_2$, (ii) R^* is the circumscribed rectangle of $R^* \cap (\bar{R}_1 \cup q \cup \bar{R}_2)$, and (iii) $R^* \cap \bar{R}_1$ is not ephemere [here, we use the assumption $10K < J$, which implies $10(l^* - 1) < D^*$].



Scheme 4.8

Let us consider the restrictions σ_1 and σ_2 of the configuration $\sigma^{(0)}$ to $\bar{R}_1$ and $\bar{R}_2$, respectively. Considering the site x and its three neighbor and nearest neighbor sites in the half-plane p, one can easily convince oneself that

$$\bar{H}(\sigma_1) + \bar{H}(q \cup \sigma_2) \geqslant \bar{H}(\xi^{(0)}) \geqslant \bar{H}(\sigma_1) + \bar{H}(q \cup \sigma_2) - 2J \qquad (4.16)$$

On the other hand,

$$H(\sigma_1) \geqslant H(\bar{Q}_1) \geqslant H(Q_1) \qquad (4.17)$$

where $\bar{Q}_1$ and Q_1 are the standard octagons inscribed into $\bar{R}_1$ and $\bar{R}_1 \cap R^*$, respectively, and

$$H(q \cup \sigma_2) \geqslant H(q \cup \bar{R}_2) \qquad (4.18)$$

In the inequality (4.17) we are using the fact that $\bar{R}_1$ is subcritical. Using the first inequality from (4.16) for $\sigma_1 = Q_1$, we get

$$\bar{H}(Q_1) + \bar{H}(q \cup \bar{R}_2) \geqslant \bar{H}(Q_1 \cup q \cup \bar{R}_2) \qquad (4.19)$$

Our aim now is to find a lower bound on the difference $H(Q_1 \cup q \cup \bar{R}_2) - H(Q(D^*, D))$. Expanding the droplet $Q_1 \cup q \cup \bar{R}_2$ to $Q(D^*, D)$, we have to compensate for the loss of all corners in $\bar{R}_2$ by the surplus area (similarly as in the proof of Lemma 4.4). Using n_h and n_v to denote the horizontal and vertical sizes of $\bar{R}_2 \backslash \bar{q}$, respectively, it suffices to notice that the area of $R^* \backslash \bar{R}_1 \cup q \cup \bar{R}_2$ is at least $\min(n_h, n_v)[D^* - (3l^* - 3)] + 2(D^* - 1)$. The last term stems from the rows containing the squares q and $\bar{q}$. On the other hand, any horizontal or vertical line through a corner of $\bar{R}_2 \backslash q'$ contains at most six corners of $\bar{R}_2$ (a bottleneck counts for four corners). As a result we get

$$\begin{aligned} &H(Q_1 \cup q \cup \bar{R}_2) - H(Q(D^*, D)) \\ &\quad \geqslant -6 \min(n_h, n_v) K + h \min(n_h, n_v)[D^* - (3l^* - 3)] + 2h(D^* - 1) - 2K \\ &\quad \geqslant 2h(D^* - 1) - 2K \end{aligned} \tag{4.20}$$

The term $-2K$ stands for two corners of $\bar{R}_2$ in the last row (on the boundary ∂R^*). In the last inequality we used the fact that $6K < h[D^* - (3l^* - 3)]$. Combining this with (4.16) and (4.19) and the bound $2J < hD^* \leqslant hD$, we get

$$\begin{aligned} &H(\xi^{(0)}) - H(Q(D^* - 1, D)) - (2J - hD) \\ &\quad \geqslant H(\xi^{(0)}) - H(Q(D^* - 1, D)) \geqslant 2(D^* - 1) h - 2K - 2J \end{aligned} \tag{4.21}$$

Thus, using the inequality

$$2(D^* - 2) h + 2K > [2(D^* - 2) + l^* - 1] h \geqslant 2D^* h > 4J \tag{4.22}$$

satisfied once $l^* \geqslant 3$ [cf. (3.2)], we get

$$H(\xi^{(0)}) - H(Q(D^* - 1, D)) \geqslant (D^* - 2) h - 2K > 2J - 4K - h \tag{4.23}$$

To get the sought inequality it remains to realize that $H(Q(D^* - 1, D)) \geqslant H(Q(D^* - 1, D^*))$ since $(D^* - 1) h < 2J$.

Consider now the case when both $\bar{R}_1$ and $\bar{R}_2$ are ephemere. For the strings not to interact, at least one of them, say $\bar{R}_2$, must be attached to the rectangle q through a rectangle $\bar{q}$, in the similar manner as above. Using the assumption $10K < J$, one can show that

$$10(l^* - 1) < D^* \tag{4.24}$$

Observing now that any ephemere string is contained in a strip of thickness $3l^* - 3$, we can conclude that, to combine to a supercritical rectangle $\tilde{R}^{(1)}$, the strips containing $\bar{R}_1$ and $\bar{R}_2 \cup q$ are necessarily orthogonal to each other. Further, there exists a $D \times D^*$ (or $D^* \times D$) rectangle R^* with $D \geqslant D^*$ such that R^* is the circumscribed rectangle of $R^* \cap (\bar{R}_1 \cup q \cup \bar{R}_2)$. The surplus area in R^* with respect to $\bar{R}_1 \cup q \cup \bar{R}_2$ is at least $(D^* - 3l^* + 3)^2$, while the number of corners in the strings $\bar{R}_1$ and $\bar{R}_2$ is at most

$2 \times 6 \times (3l^* - 3)$. Taking into account that $2K < l^*h$, $2(D^* - 3l^*) \geqslant D^*$, and $-4F(l^*) \geqslant 2(l^* - 1)^2 h$, we get the bound

$$\begin{aligned} H(\xi^{(0)}) - H(Q(D^* - 1, D)) & \\ \geqslant -2J - 12(3l^* - 3)\, K + (D^* - 3l^* + 3)^2\, h - 4F(l^*) & \\ \geqslant -2J - 6(3l^* - 3)\, l^*h + 3D^*h + (D^* - 3l^*)^2\, h + 2(l^* - 3)^2\, h & \quad (4.25) \end{aligned}$$

The sought inequality then follows from

$$-6(3l^* - 3)\, l^*h + 3D^*h + (D^* - 3l^*)^2\, h + 2(l^* - 3)^2\, h > 4J - 4K - h \quad (4.26)$$

since

$$\begin{aligned} (D^* - 3l^*)^2 + 2(l^* - 3)^2 + 2(l^* - 1) > (6l^*)^2 + (l^*)^2 > 18(l^*)^2 & \\ > 6(3l^* - 3)\, l^* & \quad (4.27) \end{aligned}$$

Next, consider the case when $\tilde{R}^{(1)}$ is subcritical. Let us introduce $\bar{R}^{(1)} = \tilde{R}^{(1)}$ if $\tilde{R}^{(1)}$ is nonephemere. Otherwise we take for $\bar{R}^{(1)}$ the string $N(C^{(0)})$ obtained as the blown up envelope of the configuration corresponding to $C^{(0)}$. Consider now $\bar{R}^{(1)}$ and all complete sets among $\bar{R}_1,\dots, \bar{R}_k$ that were not used for $C^{(0)}$ and construct from them the set of chains $\mathscr{C}_j^{(1)}$ of the first generation. A sequence $\mathscr{C}_j^{(r)}$, $r = 1,\dots, m$, of chains of following generations is obtained from it by iteration. Since the sets $\bar{R}_1,\dots, \bar{R}_k$ are mutually noninteracting, for every generation r we get a chain, say $\mathscr{C}_1^{(r)}$, consisting of a set $\bar{R}^{(r)}$ containing the site x and certain subsets of $\bar{R}_1,\dots, \bar{R}_k$, each of them interacting directly with $\bar{R}^{(r)}$. The remaining chains $\mathscr{C}_j^{(r)}$, $j = 2,\dots,$ contain each just one complete set from those among $\bar{R}_1,\dots, \bar{R}_k$ that have not appeared in $\mathscr{C}_1^{(p)}$, $p \leqslant r$, in the preceding steps. Clearly, there is only one chain in the last generation, $\mathscr{C}_1^{(m)} = \{\bar{R}^{(m)}\} \equiv \{\bar{R}\}$.

Let us consider the last set $\bar{R}^{(p)}$ among $\bar{R}^{(r)}$, $r = 1,\dots, m$, that is subcritical and take the chain $\mathscr{C}_1^{(p)}$ with the complete sets in $\mathscr{C}_1^{(p)} \backslash \bar{R}^{(p)}$ ordered in a particular way, say in lexicographic order of their left upper corner. Let us unite them, one by one in the given order, with the set $\bar{R}^{(p)}$ until the circumscribed rectangle is supercritical. Cutting off the remaining complete sets from the chain $\mathscr{C}_1^{(p)}$, we get the chain $\tilde{\mathscr{C}}_1^{(p)} \subset \mathscr{C}_1^{(p)}$. Let us use $\tilde{R}'$ to denote the last complete set that was attached to form the chain $\tilde{\mathscr{C}}_1^{(p)}$ and $\tilde{R}$ either the blown up envelope or the circumscribed rectangle of the union of complete sets from $\tilde{\mathscr{C}}_1^{(p)} \backslash \{\tilde{R}'\}$, depending on whether it is ephemere or not. Clearly, $\tilde{R}$ and $\tilde{R}'$ are subcritical interacting sets with a supercritical rectangular envelope of their union.

The energy of the configuration $\tilde{\xi}$ can be evaluated by the sum of the energies of its restriction to $\tilde{R}$ and $\tilde{R}'$, respectively, as

$$\bar{H}(\tilde{\xi}) \geqslant \bar{H}(\tilde{\xi} \upharpoonright \tilde{R}) + \bar{H}(\tilde{\xi} \upharpoonright \tilde{R}') - 2K \tag{4.28}$$

Indeed, the plusses of the original configuration $\tilde{\xi}$ are inside of the non-interacting sets $\bar{R}_1,\dots, \bar{R}_k$, and q, and thus only when $\tilde{R}'$ is touching by its corner the square q (included in $\tilde{R}$) may the additional $2K$ appear when joining the configurations $\tilde{\xi} \upharpoonright \tilde{R}$ and $\tilde{\xi} \upharpoonright \tilde{R}'$. Consider further, depending on whether $\tilde{R}$ ($\tilde{R}'$) is ephemere or not, the set $\tilde{Q}$ ($\tilde{Q}'$) defined as the set itself or its inscribed standard octagons. According to (4.8), one has

$$\bar{H}(\tilde{\xi} \upharpoonright \tilde{R}) + \bar{H}(\tilde{\xi} \upharpoonright \tilde{R}') \geqslant \bar{H}(\tilde{Q}) + \bar{H}(\tilde{Q}') \tag{4.29}$$

Let us observe now that there exists a $D^* \times D^*$ square R^* such that:

(i) It intersects both sets $\tilde{R}$ and $\tilde{R}'$ in nondegenerate sets R and R', $R = \tilde{R} \cap R^*$, and $R' = \tilde{R}' \cap R^*$.

(ii) It is the rectangular envelope of $R \cup R'$.

(iii) It contains the intersection $\tilde{R} \cap \tilde{R}'$ (if it is nonempty).

(iv) If $\tilde{R}$ ($\tilde{R}'$) is nonephemere, the same is true for R (R').

Since both sets $\tilde{R}$ and $\tilde{R}'$ are subcritical, we decrease the energy of $\tilde{Q}$ ($\tilde{Q}'$) to the energy of Q (Q') defined as $Q \cap R^*$ ($Q' \cap R^*$) for ephemere $\tilde{Q}$ ($\tilde{Q}'$) and as the inscribed standard octagon for nonephemere $\tilde{Q}$ ($\tilde{Q}'$). Hence

$$\bar{H}(\tilde{\xi}) \geqslant \bar{H}(Q) + \bar{H}(Q') - 2K \tag{4.30}$$

Thus our task is to evaluate the sum of energies of two interacting sets Q and Q' (nonephemere standard octagons or ephemere string) whose union has R^* as the rectangular envelope. Consider first the case when both Q and Q' are nonephemere. Replacing Q and Q' by the standard octagon Q^* inscribed in R^*, the sum on the right-hand side of (4.30) decreases by at least $h(D^* - 1)$, yielding (4.10). Indeed, if the rectangles R and R' just touch in the corner, the boundary has the same number of bonds as in Q^*, while the surplus area of R^* as compared with that of $R \cup R'$ is at least $D^* + 2(3l^* - 4)^2$. The sought bound follows once we realize that

$$4F(l^*) \geqslant -2(l^* - 1)^2 h - 2\eta(2l^* - 1) \geqslant -2(3l^* - 4)^2 h \tag{4.31}$$

If the rectangles R and R' are intersecting, the surplus area shrinks. However, each line of at most D^* surplus sites lost in this way is compensated by surplus 2 bonds in the joint boundary of Q and Q' as compared with the boundary of Q^*. If, on the other hand, the distance of the rectangles R and R', say in the vertical direction, is 1, there must exist at least two surplus horizontal bonds for them to interact, compensating thus the lack of two vertical bonds.

It remains to consider the case with, say, nonephemere R and ephemere R'. If R and R' just touch in the corner, the reasoning is the same as when discussing the case of supercritical $\tilde{R}^{(1)}$ above [cf. the bound (4.23)]. The cases of intersecting R and R' or of R and R' whose distance is 1 are then treated with the same modifications as when both R and R' are nonephemere.

It is easy to observe that, except for the case when a single $\bar{R}_t$ is contributing to $C^{(0)}$, the lower bounds are always sharp—at least once during the process of getting a lower bound we use a sharp bound. See, e.g., the lower bounds (4.23), (4.26), (4.27). As a result, we can conclude that the only configurations from $\partial\mathscr{A}$ on which the bound can be achieved are those from $\mathscr{P}$, the remaining ones having strictly higher energy by at least the minimal amount h. ■

5. PROOFS OF THEOREMS

Similarly as when proving the statement of Proposition 1 from the bound (3.77), we will get Theorems 1 and 2 from

$$\lim_{\beta \to \infty} P_{-\underline{1}}(\tau_{\partial\mathscr{A}} \geqslant T(\tilde{\varepsilon})) = 0 \tag{5.1}$$

which will be shown to be valid for all $\tilde{\varepsilon} > 0$ with $T(\tilde{\varepsilon}) = \exp[\beta(E^* + \tilde{\varepsilon})]$ [cf. (2.24), (2.28)]. To prove (5.1), we follow ref. 8 and define, in a similar manner as in the proof of Proposition 3.1, an event $\mathscr{E}_\sigma$ starting from an arbitrary σ in $\mathscr{A}$, taking place over an interval of time $T_1 = \exp\{\beta[E(l^* - 1) + \delta]\}$ [cf. (3.71)] and such that:

1. If $\mathscr{E}_\sigma$ takes place, then necessarily the set $\partial\mathscr{A}$ is reached (in a particular manner) before the time T_1.
2. For the probability $P(\mathscr{E}_\sigma)$ the uniform lower bound

$$\inf_{\sigma \in \mathscr{A}} P(\mathscr{E}_\sigma) \geqslant \alpha \tag{5.2}$$

holds with α such that

$$\lim_{\beta \to \infty} (1 - \alpha)^{T_2/T_1} = 0 \tag{5.3}$$

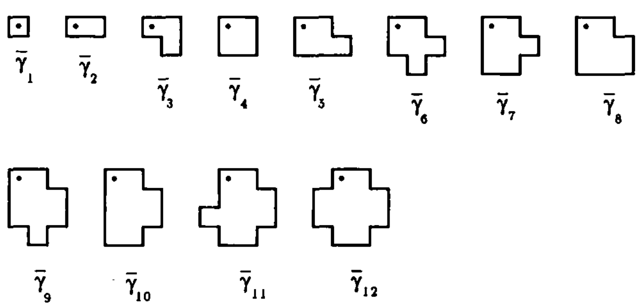
where $T_2 = T(\tilde{\varepsilon})$.

Using Eqs. (5.2) and (5.3), it follows by the strong Markov property that an attempt to reach $\partial\mathscr{A}$ not later then T_2 will be successful with high probability for large β. Indeed, it is enough to observe that after splitting T_2 into T_2/T_1 intervals of length T_1, staying inside of $\mathscr{A}$ for any of these intervals means that the event $\mathscr{E}_\sigma$ did not take place.

Once we have (5.1), we use the reversibility (Lemma 3.4) and refer to the property (iii) of the set $\mathscr{A}$ to get an upper bound on the probability of reaching $\partial\mathscr{A}$ in a configuration outside $\mathscr{P}$. Noticing that, after starting from $\mathscr{P}$, there is a finite probability to go to $+\underline{1}$ before returning to $\mathscr{A}$ (cf. Proposition 4), we get Theorems 1 and 2.

Thus, our aim is to construct an event $\mathscr{E}_\sigma$ such that (5.2) and (5.3) hold true. First, we present the idea for the construction of $\mathscr{E}_\sigma$; formal definitions will follow. We begin by recalling that $\mathscr{A}$ is defined in such a way that for all $\sigma \in \mathscr{A}$ one descends (energy decreases) to a set of noninteracting subcritical octagons in time of order T_0 [see Eq. (3.100)]. Then, with high probability, in a time shorter than $T_1 = \exp\{\beta E(L^* - 1)\}$ we reach the configuration $-\underline{1}$. The first part of $\mathscr{E}_\sigma$ refers to this shrinking phenomenon. This stage of $\mathscr{E}_\sigma$ is called *contraction* and is denoted by $\mathscr{E}^{(c)}$.

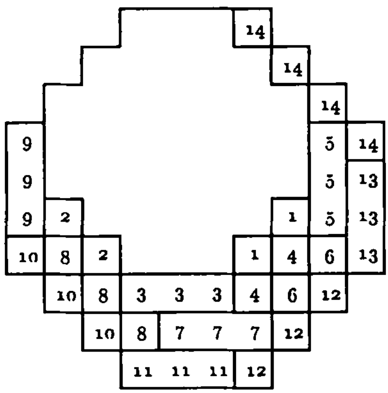
The subsequent stage consists just in staying in $-\underline{1}$ for a time of order $\exp\{\beta E(L^* - 1)\}$; it is called *waiting* and is denoted by $\mathscr{E}^{(w)}$. This (random) time spent in $-\underline{1}$ before the growth of a droplet up to the critical nucleus can be considered as a "global resistance time" as will be clear later; its introduction will lead to a gain of a factor $\exp\{\beta E(L^* - 1)\}$, due to the "temporal entropy," in our lower bound for the probability $P(\mathscr{E}_\sigma)$. [Remember that $E(L^* - 1) < 2\tilde{J} = 2J - 4K$ and the energy for creating a plus spin in a sea of minuses is $4\tilde{J} + 4K = 4J - 4K$; hence, the time for the creation of a plus spin is, with high probability, much longer than $\exp\{\beta E(L^* - 1)\}$.]



Scheme 5.1

Next, during the third stage, which we call *embryonal* and denote by

$\mathscr{E}^{(e)}$, we create sequentially the first stable regular octagon[8] (of edge 2) $Q(l=2)$ in 12 elementary (single-spin-flip) steps with increase of energy during the first 11 and with a loss of energy during the 12th. At each step the configuration will consist of a unique closed contour $\bar{\gamma}_i$, $i=1,..., 12$, as indicated in Scheme 5.1. Notice that in $\mathscr{E}_\sigma$, from $\bar{\gamma}_1$ up to the protocritical droplet, all the octagons will be centered—their upper left corner belonging to the dual lattice is located in the point $(-1/2, 1/2)$. The dot in Scheme 5.1 represents the origin.



Scheme 5.2

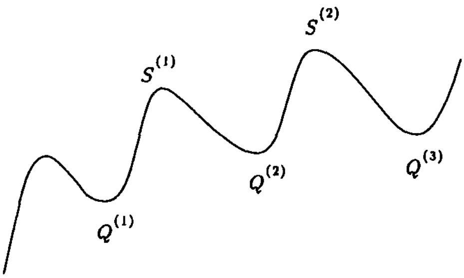
Subsequently, there is a stage called *regular*, $\mathscr{E}^{(r)}$, during which one passes through regular octagons $Q(l)$. The passage from an octagon $Q(l)$ to the following one $Q(l+1)$ is by a sequence of canonical growth (reversed to sequence of canonical cuts; cf. Lemma 3.5). Scheme 5.2 shows one particular sequence of octagons on the path from $Q(l)$ to $Q(l+1)$ (here $l=3$). Namely, the octagonal droplet $\gamma_l^{(i)}$, $i=1,..., 14$, is obtained from the preceding one, $\gamma_l^{(i-1)}$, by adding the ith edge as indicated. Notice that $\gamma_l^{(14)} \equiv \gamma_{l+1}^{(0)} \in Q(l+1)$. Further, use $S(l)$ to denote the saddle obtained by erasing from $Q(l+1)$ the last (in lexicographic order) l contiguous unit squares adjacent from the interior to its upper right oblique edge. More generally, we denote by $S_l^{(i)}$ the saddle configuration obtained by adding to $\gamma_l^{(i-1)}$ the first unit square of the ith edge.

The path, visiting 14 octagons indicated in Scheme 5.2, obtained in

[8] Recall that we identify an octagon Q with the spin configuration where the plusses are precisely the spins inside Q.

this way is *almost monotonic* path—it consists of a series of elementary transitions of the following form:

1. First, starting from an octagon $Q^{(1)}$, say $Q^{(1)} = \gamma_l^{(i-1)}$, a monotonic ascent to $S^{(1)}$ (a saddle between $Q^{(1)}$ and $Q^{(2)}$).
2. Then a descent to a configuration $Q^{(2)}$ (again an octagon), higher then $Q^{(1)}$: $H(Q^{(2)}) > H(Q^{(1)})$.
3. Another ascent to $S^{(2)}$ with $H(S^{(2)}) > H(S^{(1)})$ and so on (see Scheme 5.3).



Scheme 5.3

To get a lower bound on the probability of $\mathscr{E}^{(r)}$, we suppose that a path in $\mathscr{E}^{(r)}$ stays in the basin of attraction $\mathscr{B}(\gamma_l^{(i-1)})$ [see (3.7)] up to a random time shorter than $\exp\{\beta[H(S_l^{(i-1)}) - H(\gamma_l^{(i-1)}) + \varepsilon]\}$ with $\varepsilon > 0$ chosen sufficiently small; then it ascends to $S_l^{(i)}$ and afterward descends to $\gamma_l^{(i)}$; in the next step it stays in $\mathscr{B}(\gamma_l^{(i)})$ up to a time of order at most $\exp\{\beta[H(S_l^{(i)}) - H(\gamma_l^{(i)}) + \varepsilon]\}$, then it ascends to $S_l^{(i)}$ and so on.

The above times are called *resistance times.* Their introduction in the definition of $\mathscr{E}^{(r)}$ (and in the subsequent stages) is motivated by the necessity of exploiting the "temporal entropy" to get a correct lower bound. It turns out that by the above choice of the resistance times one gets exactly the needed factors. Notice that since the minimum of H on $\partial\mathscr{B}(\gamma_l^{(i)})$ is reached in $S_l^{(i)}$, the probability that during the time interval of the order $\exp\{\beta[H(S_l^{(i)}) - H(\gamma_l^{(i)})]\}$ the process does not leave $\mathscr{B}(\gamma_l^{(i)})$ is almost one (here we are using the reversibility of the process; see ref. 14 and Lemma 3.4 above). On the other hand, by the particular choice of a sequence of elementary transitions in the definition of $\mathscr{E}^{(r)}$, the process enters the basin of attraction $\mathscr{B}(\gamma_l^{(i)})$ through the saddle point $S_l^{(i)}$. The combination of these two facts is crucial to get a lower bound that is sufficient for the event $\mathscr{E}$ to satisfy the condition (5.3).

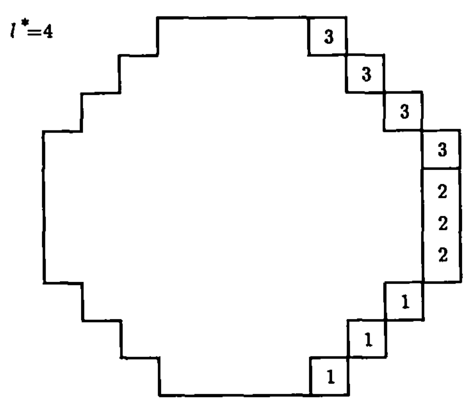
As already mentioned in the Introduction, a "local" criterion can be formulated that allows one to choose a sequence of elementary transitions to yield a correct probability estimate for the event $\mathscr{E}^{(r)}$: the passage to a successive minimum of H has to occur through a saddle whose height is exactly that of the minimum of H on the boundary of the basin of attraction of the successive minimum. We stay in the basin of attraction of this new minimum for a "typical" resistance time and then pass to the next one.

As we will see later, such a criterion can be generalized to non-almost-monotonic sequences of steps appearing in the subsequent stages of $\mathscr{E}$, provided we substitute the basin of attraction $\mathscr{B}(Q)$ of a single octagon Q by its generalization $\mathscr{D}(D_1, D_2)$ [see (3.10)].

The "resistance" inside $\mathscr{D}(D_1, D_2)$ will be against a mechanism of escape that is no longer monotonic (in energy). Actually, it is exactly the escape described in the shrinking event $\mathscr{E}^s_\sigma$ introduced in the proof of Proposition 1 [see the definition (3.95)]. In addition, the descent from a saddle [in $\partial\mathscr{D}(D_1, D_2)$] to the corresponding minimum will involve some tunneling phenomena (passing through some local saddle points).

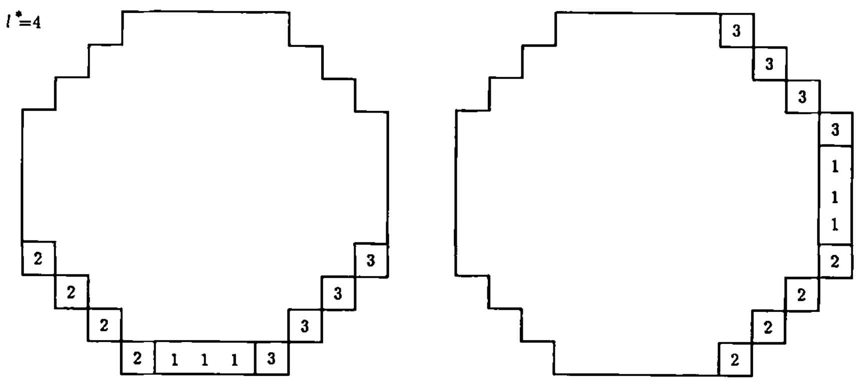
The stage $\mathscr{E}^{(r)}$ ends once we reach the octagon $Q(l^*)$. After this there is a stage that we call *transient* and denote by $\mathscr{E}^{(t)}$ during which, following a very special (not almost monotone) sequence of octagons, we reach a standard octagon with $L_1 = L_2 = l^* + 2$, namely, the first one with $\min(L_1, L_2) > l^* + 1$. To describe the "transient," which turns out to be a somewhat complicated mechanism, we need to define several particular octagons. To this end we will make repeated use of drawings.

The droplet $Q^{(0)}_{l^*,l^*}$ is the regular (and simultaneously the smallest standard) centered octagon with $l = l^*$. The configurations $\tilde{Q}^{(i)}_{l^*,l^*}$, $i = 1, 2, 3$, are the octagons obtained by adding sequentially to $Q^{(0)}_{l^*,l^*}$ the edges 1, 2, and 3, as indicated in Scheme 5.4. We notice here that $\tilde{Q}^{(1)}_{l^*,l^*}$ and $\tilde{Q}^{(2)}_{l^*,l^*}$, respectively, correspond to Q_2^* and $\tilde{Q}_1$ introduced in the proof of Proposition 2.



Scheme 5.4

The configurations $\bar{S}^{(i)}_{l^*,l^*}$, $i=1, 2, 3$, are the saddles obtained by adding to $\bar{Q}^{(i-1)}_{l^*,l^*}$ the first unit square of the ith edge. Further, let $Q^{(0)}_{l^*+1,l^*} \equiv \bar{Q}^{(3)}_{l^*,l^*}$ (= standard octagon with $L_1 = l^*+1$, $L_2 = l^*$) and let $Q^{(i)}_{l^*+1,l^*}$, $i=1,2,3$, be the octagons obtained by adding sequentially to $Q^{(i-1)}_{l^*+1,l^*}$ the edges 1, 2, and 3 as indicated in Scheme 5.5. Again, the configurations $S^{(i)}_{l^*+1,l^*}$ are the saddles obtained by adding to $Q^{(i-1)}_{l^*+1,l^*}$ the first unit square of the ith edge. Similarly, let $Q^{(0)}_{l^*+1,l^*+1} \equiv Q^{(3)}_{l^*+1,l^*}$ (= standard octagon with $L_1 = L_2 = l^*+1$) and $Q^{(i)}_{l^*+1,l^*+1}$, $i=1,2,3$, be the octagons obtained by adding sequentially to $Q^{(i-1)}_{l^*+1,l^*+1}$ the edges 1, 2, and 3 (cf. Scheme 5.5). Again, the configurations $S^{(i)}_{l^*+1,l^*+1}$ are the saddles defined in the same way as $S^{(i)}_{l^*+1,l^*}$.

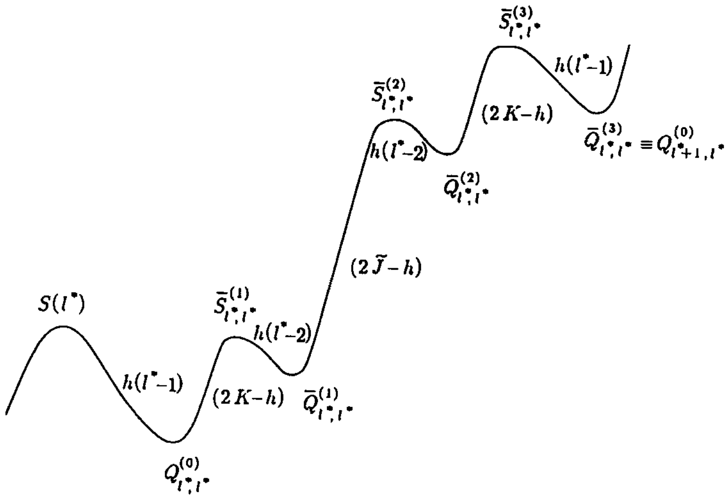


$Q^{(i)}_{l^*+1,l^*}$ $i=1,2,3$ $Q^{(i)}_{l^*+1,l^*+1}$ $i=1,2,3$

Scheme 5.5

In the first part of the event $\mathscr{E}^{(l)}$ our process will stay in $\mathscr{D}(D_1 = D_2 = l^*)$—the generalized basin of attraction of the standard (and regular) octagon $Q(D_1 = D_2 = l^*)$ that has been defined in Eq. (3.10)—for a time of order $\exp\{\beta h(l^*-1)\}$. Then, after visiting for the last time $Q^{(0)}_{l^*,l^*}$, it will pass to $\bar{Q}^{(1)}_{l^*,l^*}$ through $\bar{S}^{(1)}_{l^*,l^*}$, staying subsequently in $\mathscr{B}(\bar{Q}^{(1)}_{l^*,l^*})$ (the true basin of attraction of $\bar{Q}^{(1)}_{l^*,l^*}$) for a time of order $\exp\{\beta h(l^*-2)\}$. Then it jumps to $\bar{Q}^{(2)}_{l^*,l^*}$, passing through $\bar{S}^{(2)}_{l^*,l^*}$; it stays in $\mathscr{B}(\bar{Q}^{(2)}_{l^*,l^*})$ for a time of order $\exp\{\beta h(l^*-2)\}$ and then it passes again to $\bar{Q}^{(3)}_{l^*,l^*}$ through $\bar{S}^{(3)}_{l^*,l^*}$. [Notice that when we say "the process stays in a certain set of configurations for a time of order $\exp\{\beta\Delta\}$" we actually mean "it stays there for a random time shorter than $\exp\{\beta(\Delta+\varepsilon)\}$ with a suitable, sufficiently small, positive ε."]

This first part of $\mathscr{E}^{(t)}$ can be better understood by observing the landscape of the energy depicted in Scheme 5.6.



Scheme 5.6

The second and third parts of $\mathscr{E}^{(t)}$, consisting of the transitions from the octagons $Q^{(0)}_{l^*+1,l^*}$ to $Q^{(0)}_{l^*+1,l^*+1}$ and from $Q^{(0)}_{l^*+1,l^*+1}$ to $Q^{(0)}_{l^*+2,l^*+1}$, respectively, are very similar.

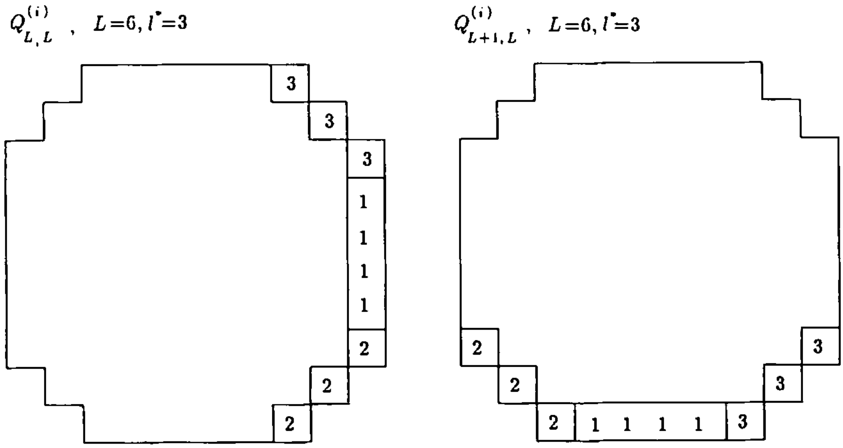
Namely, we stay in $\mathscr{B}(Q^{(0)}_{l^*+1,l^*})$ for a time of order $\exp\{\beta h(l^*-1)\}$, then pass to $Q^{(1)}_{l^*+1,l^*}$ through $S^{(1)}_{l^*+1,l^*}$ and stay in $\mathscr{B}(Q^{(1)}_{l^*+1,l^*})$ for a time $\exp\{\beta h(l^*-2)\}$; after that we go to $Q^{(2)}_{l^*+1,l^*}$ passing through $S^{(2)}_{l^*+1,l^*}$ and after staying in $\mathscr{B}(Q^{(2)}_{l^*+1,l^*})$ for a time $\exp(2K-h)\}$ we pass through $S^{(3)}_{l^*+1,l^*}$ to $Q^{(3)}_{l^*+1,l^*+1}$.

After reaching the octagon $Q^{(0)}_{l^*+2,l^*+1}$, the last part of $\mathscr{E}^{(t)}$ starts; it consists of a transition to $Q^{(0)}_{l^*+2,l^*+2}$ via a mechanism that will be repeated several times during the subsequent stage of $\mathscr{E}$ that we call *standard* and denote by $\mathscr{E}^{(s)}$. In other words, the last part of $\mathscr{E}^{(t)}$ can be considered also as the first part of $\mathscr{E}^{(s)}$.

During $\mathscr{E}^{(s)}$ our process will visit a sequence of growing standard octagons inscribed in squares or almost squares of the form

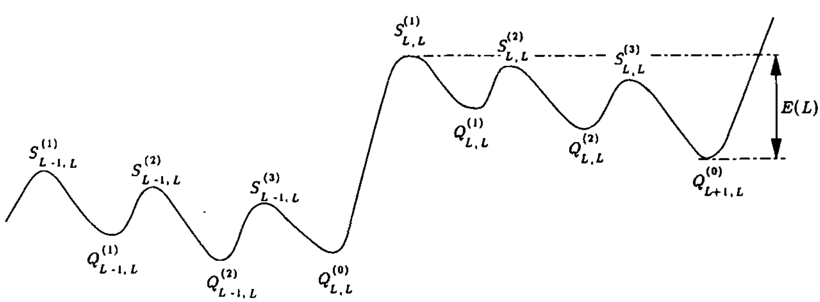
$$L_1,\ L_2 \equiv L,\ L+1 \to L+1,\ L \to L+1,\ L+1 \to L+2,\ L+1,...$$

from $L_1 = l^*+2$, $L_2 = l^*+1$ up to $L_1 = L^*$, $L_2 = L^*-1$ [cf. (2.17)]. These transitions are performed via a canonical (not almost monotonic) sequence of (not standard) octagons. Consider a standard octagon with $L_1 = L_2 = L$, $l^*+2 \leqslant L \leqslant L^*-1$ and with upper left corner in $(-1/2, 1/2)$; call it $Q^{(0)}_{L,L}$, $i=1, 2, 3$, be the octagons obtained by adding sequentially to $Q^{(i-1)}_{L,L}$ the edges 1, 2, and 3 as indicated in Scheme 5.7. Similarly, we call $Q^{(0)}_{L+1,L} \equiv Q^{(3)}_{L,L}$ the standard octagon with $L_1 = L+1$, $L_2 = L$, $l^* + 1 \leqslant L \leqslant L^* - 1$, and with left upper corner in $(-1/2, 1/2)$; $Q^{(i)}_{L+1,L}$, $i = 1, 2, 3$, are the octagons obtained by adding sequentially to $Q^{(i-1)}_{L+1,L}$ the edges 1, 2, 3 as indicated in Scheme 5.7. The configurations $S^{(i)}_{L_1,L_2}$, $i = 1, 2, 3$, are the saddles obtained from $Q^{(i-1)}_{L,L_2}$ by adding the first unit square to the ith edge.



Scheme 5.7

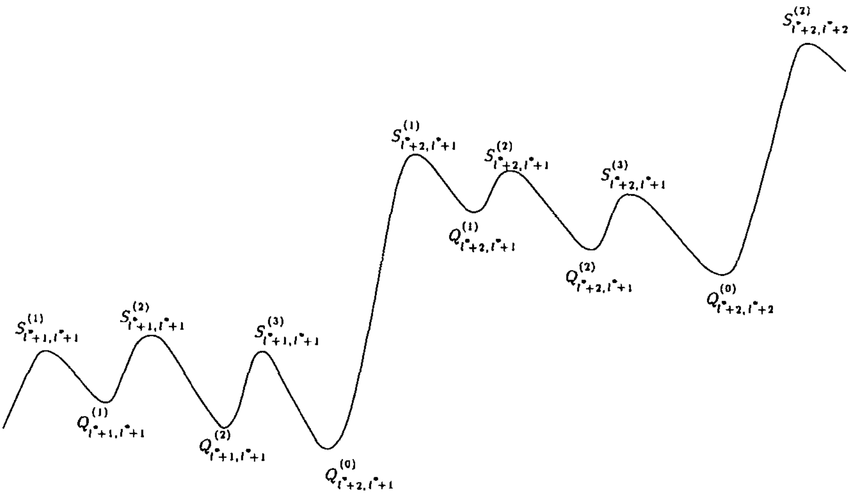
The canonical mechanism to pass from a standard octagon, say $Q^{(0)}_{L,L}$, to the following one in the sequence, $Q^{(0)}_{L+1,L}$, is as follows. We stay in $\mathscr{D}(D_1, D_2)$, $D_1 = L + 2(l^* - 1)$, $D_2 = L + 2(l^* - 1)$, for the time $\exp\{\beta E(L)\}$ [see Eqs. (3.71) and (3.72)]; then we jump to $S^{(1)}_{L,L}$ and then in a time $\sim\exp\{(2K - h)\beta\}$ we pass to the next standard octagon $Q^{(0)}_{L+1,L}$. This transition from $S^{(1)}_{L,L}$ to $Q^{(0)}_{L+1,L}$ is not a purely downhill path, but involves two tunnelings. The case of the transition $Q^{(0)}_{L+1,L} \to Q^{(0)}_{L+1,L+1}$ is completely analogous: in general the resistance time in $\mathscr{D}(D_1, D_2)$ with



Scheme 5.8

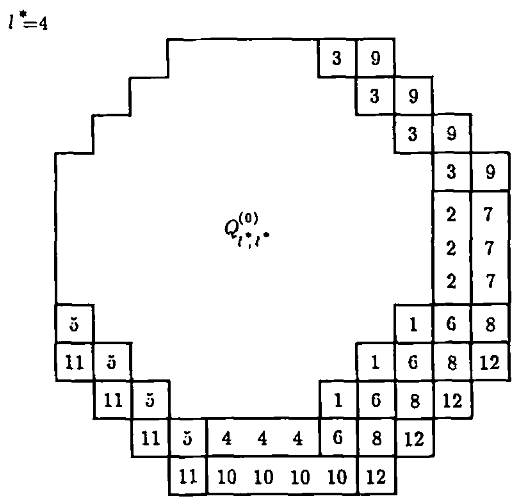
$D_1 = L_1 + 2(l^* - 1)$ and $D_2 = L_2 + 2(l^* - 1)$ is $\exp\{\beta E(L_1 \wedge L_2)\}$. A transition from $Q_{L,L}$ to $Q_{L+1,L}$ for $L \geqslant l^* + 2$ is represented in Scheme 5.8.

A similar picture describes the transition $Q^{(0)}_{L+1,L} \to Q^{(0)}_{L+1,L+1}$. In Scheme 5.9 we represent the transition from $Q^{(0)}_{l^*+2,l^*+1}$ to $Q^{(0)}_{l^*+2,l^*+2}$, namely the last part of the transient event.

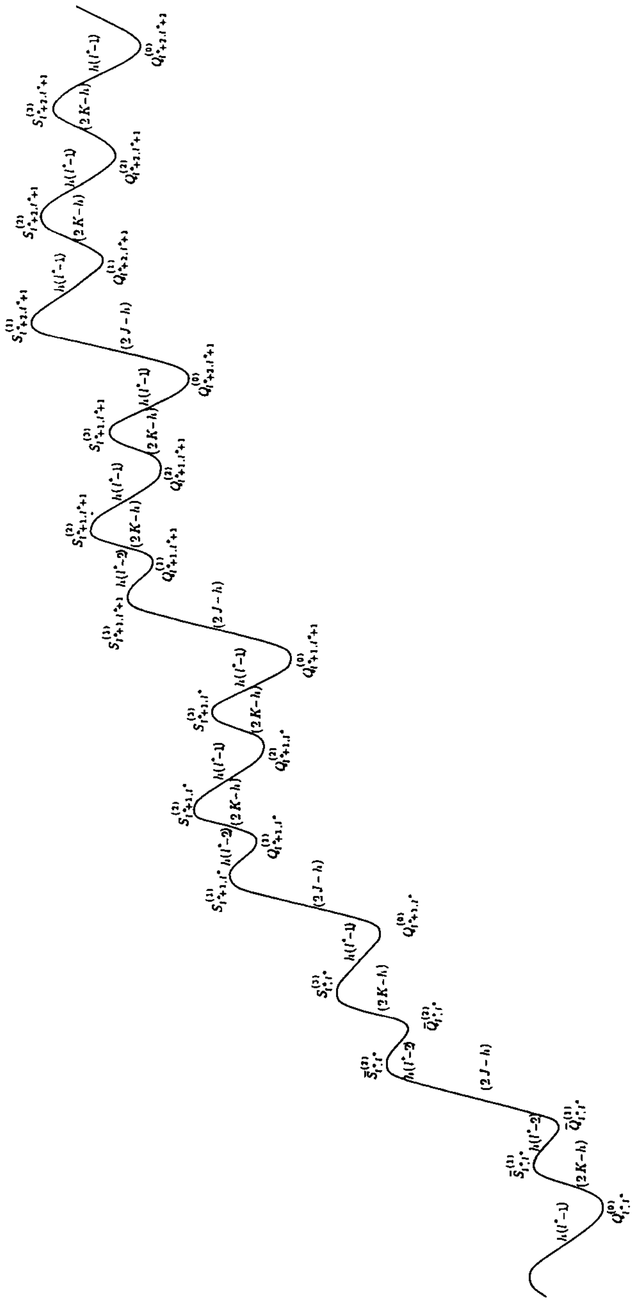


Scheme 5.9

We summarize the description of the transient event $\mathscr{E}^{(t)}$ in Schemes 5.10 and 5.11.



Scheme 5.10



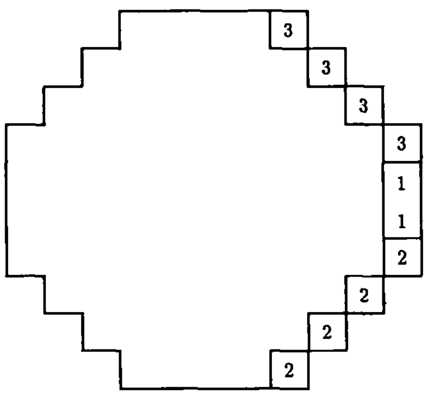
Scheme 5.11

Before passing to a formal description of the event $\mathscr{E}$, we would like to make several additional observations concerning the events $\mathscr{E}^{(t)}$ and $\mathscr{E}^{(s)}$.

1. The transition $\gamma^{(13)}_{l^*-1} \to S(l^*) \to Q^{(0)}_{l^*,l^*} \to \bar{S}^{(1)}_{l^*,l^*} \to \cdots$ is not almost monotonic.

2. The sequence $\bar{Q}^{(1)}_{l^*,l^*}$, $\bar{Q}^{(2)}_{l^*,l^*}$, $\bar{Q}^{(3)}_{l^*,l^*}$ is not the analog of $Q^{(1)}_{l^*+1,l^*}$, $Q^{(2)}_{l^*+1,l^*}$, $Q^{(3)}_{l^*+1,l^*}$ or, more generally, of $Q^{(1)}_{L_1,L_2}$, $Q^{(2)}_{L_1,L_2}$, $Q^{(3)}_{L_1,L_2}$ for the subsequent values of L_1, L_2. This part of the event $\mathscr{E}^{(t)}$ provides a good example to clarify the local criterion we referred to before. In the sequence $Q^{(0)}_{l^*,l^*}$, $\bar{S}^{(1)}_{l^*,l^*}$, $\bar{Q}^{(1)}_{l^*,l^*}$, and $\bar{S}^{(2)}_{l^*,l^*}$ we always respect the condition of passing through a saddle S such that $H(S)$ is the minimum in the boundary of the (possibly generalized) subsequent basin of attraction.

To clarify better this point, let us suppose that we defined the first part of $\mathscr{E}^{(t)}$ concerning the transition $Q^{(0)}_{l^*,l^*} \to Q^{(0)}_{l^*+1,l^*}$ in a different way. Namely, instead of $\bar{Q}^{(i)}_{l^*,l^*}$ we take the sequence of octagons $Q^{(i)}_{l^*,l^*}$, $i=1, 2$, and 3, defined in the usual way according to Scheme 5.12. Notice that $\bar{Q}^{(1)}_{l^*,l^*} \neq Q^{(1)}_{l^*,l^*}$, whereas $\bar{Q}^{(2)}_{l^*,l^*} = Q^{(2)}_{l^*,l^*}$, $\bar{Q}^{(3)}_{l^*,l^*} = Q^{(3)}_{l^*,l^*} \equiv Q^{(0)}_{l^*+1,l^*}$, and $Q^{(i)}_{l^*,l^*}$ would be the analog for $L_1 = L_2 = l^*$ of our $Q^{(i)}_{L_1,L_2}$ as they are defined for the subsequent L_1, L_2's.

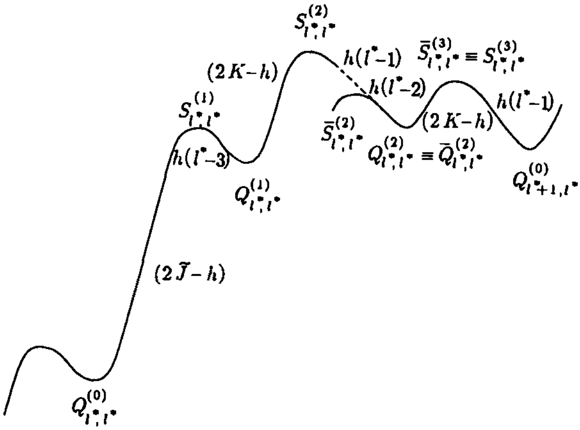


Scheme 5.12

If we define $S^{(i)}_{l^*,l^*}$ as the saddle obtained by adding to $Q^{(i-1)}_{l^*,l^*}$ the first unit square in the ith edge, we can represent the landscape of energy for this different transition as indicated in Scheme 5.13.

We observe that in the mechanism described in Scheme 5.13 what gets wrong with respect to our criterion is the transition $Q^{(1)}_{l^*,l^*} \to S^{(2)}_{l^*,l^*} \to Q^{(2)}_{l^*,l^*}$, since the minimum of H in $\partial\mathscr{D}(Q^{(2)}_{l^*,l^*})$ is reached in $\bar{S}^{(2)}_{l^*,l^*}$ and not in $S^{(2)}_{l^*,l^*}$.

3. In the sequence $Q^{(0)}_{l^*+1,l^*} \to Q^{(0)}_{l^*+1,l^*+1} \to Q^{(0)}_{l^*+2,l^*+2}$ we could have followed another path without violating our local criterion. This is a



Scheme 5.13

consequence of a degeneracy of the minimum of H in $\mathscr{D}(D_1 = 3l^* - 1, D_2 = 3l^* - 1)$ and in $\mathscr{D}(D_1 = 3l^*, D_2 = 3l^* - 1)$; as we already noticed in the proof of Proposition 2, this minimum is reached for $L_1 \wedge L_2 = l^* + 1$, both in $\tilde{S}_2$ and in $\hat{S}_2$, where $\tilde{S}_2$ and $\hat{S}_2$ are saddles defined during the proof of Proposition 1 and represented in Schemes 3.8 and 3.9, respectively. To be more precise, consider the sequence

$$\bar{Q}^{(1)}_{l^*+1,l^*},\ \bar{Q}^{(2)}_{l^*+1,l^*} \equiv Q^{(2)}_{l^*+1,l^*},\ \bar{Q}^{(3)}_{l^*+1,l^*} = Q^{(3)}_{l^*+1,l^*},\ \bar{S}^{(1)}_{l^*+1,l^*},\ \bar{S}^{(2)}_{l^*+1,l^*},\ \bar{S}^{(3)}_{l^*+1,l^*}$$

similarly to $\bar{Q}^{(1)}_{l^*,l^*}$, $\bar{Q}^{(2)}_{l^*,l^*}$, $\bar{Q}^{(3)}_{l^*,l^*}$, $\bar{S}^{(1)}_{l^*,l^*}$, $\bar{S}^{(2)}_{l^*,l^*}$, and $\bar{S}^{(3)}_{l^*,l^*}$ (see Scheme 5.4). Then, for $D_1 = 3l^* - 1 = D_2$, the configuration $\bar{S}^{(2)}_{l^*,l^*}$ coincides with $\hat{S}_2$ (see Scheme 3.9) and

$$\min_{\sigma \in \partial B(D_1, D_2)} H(\sigma) = H(\bar{S}^{(2)}_{l^*+1,l^*}) = H(\hat{S}^{(2)}_{l^*+1,l^*}) \tag{5.4}$$

A similar statement is true for $D_1 = 3l^*$, $D_2 = 3l^* - 1$.

4. Our mechanism of growth is, in all the stages, exactly the reverse of the best mechanism of shrinking described in the proof of Propositions 1–3.

To clarify the relation between the notation we used in Propositions 1–3 to describe the shrinking phenomenon (following the drift) and the present construction for describing the growth up to the critical size (against the drift), consider, for example, a standard octagon $\tilde{Q}_0$ with $L_2 = L + 1$, $L_1 = L$. We recall that $\tilde{Q}_3$ is again a standard octagon with $L_1 = L_2 = L$. We have

$$Q^{(3)}_{L,L-1} \equiv Q^{(0)}_{L,L} = \tilde{Q}_3$$

$$Q^{(1)}_{L,L} = \tilde{Q}_2, \qquad Q^{(2)}_{L,L} = \tilde{Q}_1, \qquad Q^{(3)}_{L,L} = \tilde{Q}_0$$

5. Looking at Scheme 5.10, one easily realizes that the relative heights in energy of $S^{(1)}_{L_1,L_2}$, $S^{(2)}_{L_1,L_2}$, and $S^{(3)}_{L_1,L_2}$, in the transitions from L_1, $L_2 \to L'_1$, L'_2 in $\mathscr{E}^{(l)}$, namely, in the transitions l^*, $l^* \to l^*+1$, $l^* \to l^*+1$, $l^*+1 \to l^*+2$, $l^*+1 \to l^*+2$, l^*+2, change according to the value of $L'_1 \wedge L'_2$.

For $L'_1 \wedge L'_2 = l^*$, the saddle $S^{(3)}$ is the highest one.

For $L'_1 \wedge L'_2 = l^*+1$, the saddle $S^{(2)}$ is the highest one.

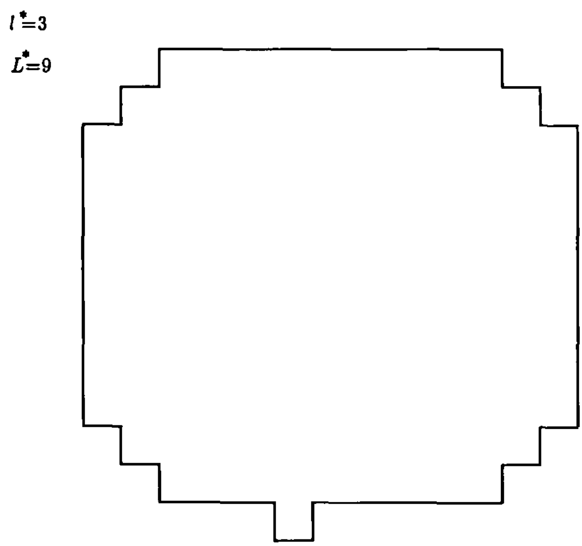
For $L'_1 \wedge L'_2 = l^*+2$ (or larger) the saddle $S^{(1)}$ is the highest one.

In any case the differences in height between the saddles (that change sign passing from $L'_1 \wedge L'_2 = l^*$ to $L'_1 \wedge L'_2 = l^*+2$) are, in absolute value, of order ηh with $\eta < 1$ defined in Eq. (2.18).

6. The introduction of the generalized basin $\mathscr{D}(D_1, D_2)$ in place of the usual $\mathscr{B}(Q)$ (that was needed in $\mathscr{E}^{(l)}$ and $\mathscr{E}^{(s)}$) is strictly related to the presence of a non-almost-monotonic path.

The next to the last transition in the standard stage ends with the formation of a standard octagon with $L_1 = L^*$ and $L_2 = L^* - 1$.

The very last transition is then just a flip of a minus spin adjacent to a long coordinate edge in the standard octagon with $L_1 = L^*$ and $L_2 = L^* - 1$ (namely, the creation of a unit-square protuberance adjacent to that edge). In this way we form a global saddle ("protocritical") droplet in $\mathscr{P}$ (see ref. 11) and enter into $\partial\mathscr{A}$ (see Scheme 5.14).



Scheme 5.14

To conclude our preliminary discussion, we want to say that the property of $Q(D^*, D^*)$ of being the minimal supercritical standard octagon can be expressed by the fact that L^* is the minimal value of L for which

$$H(S_{L,L}^{(1)}) - H(Q_{L+1,L}) = E(L) > 2\tilde{J} - h$$

Remember that during the growth up to the protocritical droplet, described by $\mathscr{E}^{(s)}$, we had

$$H(S_{L,L}^{(1)}) - H(Q_{L+1,L}) < 2\tilde{J} - h$$

as indicated in Scheme 5.8.

Let us now start with the formal definition of $\mathscr{E}$. Given $\sigma \in \mathscr{A}$ and $t_c \in \mathbb{N}$ to be fixed later, we define

$$\mathscr{E}_{\sigma, t_c}^{(c)} = \{\sigma_0 = \sigma, \tau_{-\underline{1}} = t_c\} \tag{5.5}$$

Given $t_w \in \mathbb{N}$, we set

$$\mathscr{E}_{t_w}^{(w)} = \{\sigma_t = -\underline{1}, 0 \leqslant t \leqslant t_w\} \tag{5.6}$$

Now consider the sequence of clusters $\bar{\gamma}_1, \ldots, \bar{\gamma}_{12}$ specified in Scheme 5.1. We define

$$\mathscr{E}^{(e)} = \{\sigma_0 = -\underline{1}, \sigma_1 = \bar{\gamma}_1, \ldots, \sigma_{12} = \bar{\gamma}_{12}\} \tag{5.7}$$

We recall that the energy is strictly increasing from -1 to $\bar{\gamma}_{12}$; indeed, we have

$$\begin{aligned}
&H(\bar{\gamma}_1) - H(-\underline{1}) = 4J - 4K - h, \quad && H(\bar{\gamma}_2) - H(\bar{\gamma}_1) = 2J - h \\
&H(\bar{\gamma}_3) - H(\bar{\gamma}_2) = 2J - h - 2K, && H(\bar{\gamma}_4) - H(\bar{\gamma}_3) = -h + 2K \\
&H(\bar{\gamma}_5) - H(\bar{\gamma}_4) = 2J - 2K - h, && H(\bar{\gamma}_6) - H(\bar{\gamma}_5) = 2J - 4K - h \\
&H(\bar{\gamma}_7) - H(\bar{\gamma}_6) = 2K - h, && H(\bar{\gamma}_8) - H(\bar{\gamma}_7) = 2K - h \\
&H(\bar{\gamma}_9) - H(\bar{\gamma}_8) = 2J - 4K - h, && H(\bar{\gamma}_{10}) - H(\bar{\gamma}_9) = 2K - h \\
&H(\bar{\gamma}_{11}) - H(\bar{\gamma}_{10}) = 2J - 4K - h
\end{aligned} \tag{5.8}$$

To present explicit definitions for the subsequent stages $\mathscr{E}^{(r)}$, $\mathscr{E}^{(t)}$, $\mathscr{E}^{(s)}$ of the event $\mathscr{E}_\sigma$, we would like to use the general setup described in Section 3 based on the introduction of a set of auxiliary Markov chains.

Thus, we have to specify an integer N, a sequence of octagons $Q_1, \ldots, Q_N$, connected sets $B_1, \ldots, B_N$, saddles $S_2, \ldots, S_{N+1}$, with $S_i \in \partial B_i \cap B_{i+1}$, $i = 1, \ldots, N$, $S_{N+1} \in \partial B_N$, as well as resistance times $\bar{t}_u^i$ and "descent" times $\bar{t}_d^i$.

We begin by saying that part of the octagons $Q_1,\dots, Q_N$ entering in our construction will be standard [see Eq. (2.17)]. The other ones will be non-standard (some l_i will differ from l^*); further, the sets B_i, $i=1,\dots, N$ will be either (1) the basin of attraction $\mathscr{B}(Q_i)$ of Q_i [see Eq. (3.7)] if Q_i is not standard, or (2) the domain of attraction $\mathscr{D}(D_1, D_2)$ [see Eq. (3.10)] if $Q=Q(D_1, D_2)$ is a standard octagon.

At the end of our construction we will consider the set $\bar{\mathscr{F}} \cap \mathscr{G}$ [cf. Eqs. (3.57), (3.54), (3.56), (3.48), and (3.53)]. It will contain the regular, the transient, and the standard stages. We have, with obvious meaning of the symbols,

$$N = N_r + N_t + N_s$$

$$N_r = 14(l^*-2), \qquad N_t = 7, \qquad N_r = 2[L^* - (l^*+2)] \tag{5.9}$$

and

$$\begin{aligned} &Q_1, S_2, Q_2, S_3,\dots, Q_N, S_{N+1} \\ &\quad \equiv \gamma_2^0, S_2^1,\dots, \gamma_2^{13}, S_2^{14}, \gamma_3^0,\dots, \gamma_{l^*-1}^{13}, S_{l^*-1}^{14}, Q_{l^*,l^*}^{(0)}, \\ &\quad\quad \bar{S}_{l^*,l^*}^{(2)}, \bar{Q}_{l^*,l^*}^{(2)}, S_{l^*,l^*}^{(3)}, Q_{l^*+1,l^*}^{(0)}, S_{l^*+1,l^*}^{(1)}, Q_{l^*+1,l^*}^{(1)}, S_{l^*+1,l^*}^{(2)}, \\ &\quad\quad Q_{l^*+1,l^*+1}^{(0)}, S_{l^*+1,l^*+1}^{(1)}, Q_{l^*+1,l^*+1}^{(1)}, S_{l^*+1,l^*+1}^{(2)}, Q_{l^*+2,l^*+1}^{(0)}, \\ &\quad\quad S_{l^*+2,l^*+1}^{(1)}, Q_{l^*+2,l^*+2}^{(0)}, S_{l^*+2,l^*+2}^{(1)},\dots, S_{L^*-1,L^*-1}^{(1)}, \\ &\quad\quad Q_{L^*,L^*-1}^{(0)}, S_{L^*,L^*-1}^{(1)} \end{aligned} \tag{5.10}$$

Now we specify the times $\bar{t}_u^i$, $\bar{t}_d^i$ corresponding to the different stages: regular, transient, and standard.

The times $\bar{t}_u^i$, $\bar{t}_d^i$ for the regular stages are denoted by $\bar{t}_u^{j,l}$, $\bar{t}_d^{j,l}$.

They correspond, respectively, to the transitions

$$\gamma_l^{(j)} \to S_l^{j+1}, \quad j=0,\dots, 13, \quad S_l^{j+1} \to \gamma_l^{(j+1)}, \quad j=0,\dots, 12, \quad S_l^{14} \to \gamma_{l+1}^{(0)}$$

for l: $2 \leqslant l \leqslant l^*-1$. They are given by

$$\begin{aligned} &\bar{t}_u^{j,l} = \exp\{\beta[h(l-1)+\delta]\} \qquad &&\text{for} \quad j \neq 1, 2, 4, 6 \\ &\bar{t}_u^{j,l} = \exp\{\beta[h(l-2)+\delta]\} \qquad &&\text{for} \quad j = 1, 2, 4, 6 \\ &\bar{t}_d^{j,l} = \exp\{\beta\delta\} \end{aligned} \tag{5.11}$$

with δ to be chosen later.

For the transient stage we have

$$\bar{t}_u^j = \exp\{\beta[h(l^*-2)+\delta]\}$$

for the transitions

$$\bar{Q}^{(2)}_{l^*,l^*} \to S^{(3)}_{l^*,l^*}, \qquad Q^{(1)}_{l^*+1,l^*} \to S^{(2)}_{l^*+1,l^*}, \qquad Q^{(1)}_{l^*+1,l^*+1} \to S^{(2)}_{l^*+1,l^*+1}$$

further,

$$\bar{t}^j_u = \exp\{\beta[h(l^*-1)+\delta]\}$$

for the transitions

$$Q^{(0)}_{l^*,l^*} \to \bar{S}^{(2)}_{l^*,l^*}, \qquad Q^{(0)}_{l^*+1,l^*} \to S^{(1)}_{l^*+1,l^*}$$

as well as

$$\bar{t}^j_u = \exp\{\beta[2h(l^*-1)-2(K-h)]+\delta)\}$$

for

$$Q^{(0)}_{l^*+1,l^*+1} \to S^{(1)}_{l^*+1,l^*+1}, \qquad Q^{(0)}_{l^*+2,l^*+1} \to S^{(1)}_{l^*+2,l^*+1}$$

and

$$\bar{t}^j_u = \exp\{\beta[E(l^*+2)+\delta]\}$$

for

$$Q^{(0)}_{l^*+2,l^*+1} \to S^{(1)}_{l^*+2,l^*+2}$$

Moreover, we take

$$\bar{t}^j_d = \exp(\beta\delta)$$

for the transitions

$$S(l^*) \to Q^{(0)}_{l^*,l^*}, \qquad \bar{S}^{(2)}_{l^*,l^*} \to \bar{Q}^{(2)}_{l^*,l^*}, \qquad S^{(3)}_{l^*,l^*} \to Q^{(0)}_{l^*+1,l^*}$$
$$S^{(1)}_{l^*+1,l^*} \to Q^{(1)}_{l^*+1,l^*}, \qquad S^{(1)}_{l^*+1,l^*+1} \to Q^{(1)}_{l^*+1,l^*+1}$$

For all the other transitions of the transient stage we take

$$\bar{t}^j_d = \exp\{\beta(2K-h+\delta)\}$$

and for any transition

$$Q^{(0)}_{L_1,L_2} \to S^{(1)}_{L_1,L_2}$$

of the standard stage,

$$\bar{t}^j_u = \exp\{\beta[E(L_1 \wedge L_2)+\delta]\}$$

Finally, for every *descent* time of the standard stage we take

$$\bar{t}_d^j = \exp\{\beta(2K - h + \delta)\}$$

Now we are ready to present the definition of our event $\mathscr{E}_\sigma$,

$$\begin{aligned}\mathscr{E}_\sigma &= (\bar{\mathscr{E}}_{t_c}^{(c)}; \bar{\mathscr{E}}_{t_w}^{(w)}; \mathscr{E}^{(e)}; (\bar{\mathscr{F}} \cap \mathscr{G}))\\ &= \bigcup_{t_c=1}^{\bar{t}_c} \bigcup_{t_w=1}^{\bar{t}_w} \mathscr{E}_{t_c}^{(c)} \cap T_{t_c} \mathscr{E}_{t_w}^{(w)} \cap T_{t_c+t_w} \mathscr{E}^{(e)} \cap T_{t_c+t_w+12}(\bar{\mathscr{F}} \cap \mathscr{G})\end{aligned} \tag{5.12}$$

We repeat that $\bar{\mathscr{F}} \cap \mathscr{G}$ is the event corresponding to the regular, transient, and standard stages defined as in Eq. (3.57) using the previously defined Q's, B's, S's [see Eq. (5.10)] and the corresponding times $\bar{t}_j^u, \bar{t}_j^d$. It is immediate to verify inequality (3.62) in our case. Now, to get the basic estimate given by inequality (3.67), namely, in our case,

$$P(\bar{\mathscr{F}}) \geqslant \exp\{-\beta[H(\mathscr{P}) - H(S_1) + \varepsilon]\} \tag{5.13}$$

we first need to verify (3.66). For every *downhill* transition, namely for every transition of the regular case, as well as for the *descent* transitions of the transient case corresponding to $\bar{t}_d^j = \exp(\beta\delta)$, this is an immediate consequence of the inequality

$$\left(\frac{1}{|\Lambda|}\right)^{T_0} > \exp(-\varepsilon\beta)$$

valid for every $\varepsilon > 0$ and β sufficiently large [see (3.100)].

For the transitions

$$S_{l^*+1,l^*}^{(2)} \to Q_{l^*+1,l^*+1}^{(0)}, \quad S_{l^*+1,l^*+1}^{(2)} \to Q_{l^*+2,l^*+1}^{(0)}, \quad S_{l^*+2,l^*+1}^{(1)} \to Q_{l^*+2,l^*+2}^{(0)}$$

of the transient stage, as well as for every transition of the standard case, we notice that from S_j in one step, with probability larger that $1/|\Lambda|$ we go to B_{j+1}. Then Eq. (3.66) is an immediate consequence of the argument of the proof of Proposition 1.

The inequality (3.65) is very easy to deduce by remarking that $Q_j \to S_{j+1}$ is always an uphill single-spin-flip transition with the exception of the first part of the transient stage, $Q_{l^*,l^*}^{(0)} \to \bar{S}_{l^*,l^*}^{(2)}$. In this last case it is easy to prove Eq. (3.65) by using a trial event with the resistance time of the order $\exp\{\beta[h(l^* - 2) + \delta]\}$ in $\bar{Q}_{l^*,l^*}^{(1)}$ exploited in the usual way. We leave the details to the reader.

For all the other cases that, we repeat, are single-spin-flip uphill transitions, the lower bound given by (3.65) is immediate.

To get the equation of the form (3.59) we proceed as in the proof of Proposition 1. Namely, we prove condition (C1)

$$\text{there exists} \quad C > 0 \quad \text{such that} \quad P(\mathscr{G}^c) < \exp(-e^{C\beta}) \qquad (5.14)$$

by introducing, for every $j = 1,..., N$, certain events $\hat{\mathscr{E}}_\sigma^j$ and times $\tilde{t}_u^j$ satisfying conditions (3.98) and (3.99).

To this end, we observe that for every nonstandard (subcritical) octagon Q_j appearing in $\mathscr{G}$ the corresponding B_j is just the usual basin of attraction and we can introduce the event $\hat{\mathscr{E}}_\sigma^j$ in the following way:

1. First, we pass to Q_j via a descent path in a time of order T_0. [For every $\varepsilon > 0$ and β sufficiently large the corresponding probability is larger than $\exp(-\varepsilon\beta)$.]

2. Then we pass to the minimal saddle S in ∂B_j (corresponding to the shrinking mechanism, since Q_j is subcritical) by a sequence of corner erosions.

The corresponding probability estimate is like (3.65) with a proper choice of ε.

Otherwise, for the cases of standard octagons (appearing both in the transient and in the standard stages) we take $B_j = \mathscr{D}(L_1 + 2(l^* - 1), L_2 + 2(l^* - 1))$ for the domain of attraction. The time $\tilde{t}_u^j$ can be taken as $\exp\{\beta[E(L_1 \wedge L_2) + \varepsilon]\}$ and the escape event $\hat{\mathscr{E}}_\sigma^j$ is just the shrinking event $\mathscr{E}_\sigma^s$ constructed in the proof of Proposition 1.

Now, it is easy to see that, for every sufficiently small $\varepsilon > 0$ and β sufficiently large, if

$$\bar{t}_w = \exp\{\beta[E(L^* - 1) + \delta]\}, \qquad \delta > 0$$

then

$$P(\bar{\mathscr{E}}_{\bar{t}_w}^{(w)}; \mathscr{E}^e) \geq \exp\{\beta[E(L^* - 1) - H(\bar{\gamma}_{11}) + H(-\underline{1}) - \varepsilon]\} \qquad (5.15)$$

Moreover, from Propositions 1–3 it follows that for every sufficiently small ε and β sufficiently large, once

$$\bar{t}_c = \bar{t}_w = \exp\{\beta[E(L^* - 1) + \delta]\}$$

then

$$P(\bar{\mathscr{E}}_{i_c}) \geqslant \exp(-\beta\varepsilon) \tag{5.16}$$

From (5.12)–(5.14), (3.67), (5.15), and (5.16) it follows that

$$P(\mathscr{E}_\sigma) \geqslant \exp\{\beta[E(L^*-1)+H(\mathscr{P})-H(-\underline{1})-\varepsilon]\}$$

for all sufficiently small ε and β sufficiently large.

Finally, from (5.2), (5.3), and (5.15) we get (5.1).

Describing the event $\mathscr{E}$, we actually defined an *ε-typical path* appearing in the statement of Theorem 3.

The only difference is that while in the definition of $\mathscr{E}_\sigma$ we considered a very particular sequence of configurations (for example, all concerned octagons are centered), defining the set $\mathscr{U}_\varepsilon$ of all ε-typical paths we can be slightly more flexible and allow also droplets of different positions and orientations.

Namely, an ε-typical path describes the typical way followed by our process starting from $-\underline{1}$ to form a critical nucleus and then to go to $+\underline{1}$. It contains, in particular, the stages that we have described when defining our trial event $\mathscr{E}_\sigma$ except for the initial *contraction* and *waiting* stages. In other words an ε-typical path will pass, initially, through the *embryonal*, *regular*, *transient*, and *standard* stages, spending, in the appropriate basins or domains of attraction, suitable intervals of time (*resistance times*).

Then, after reaching the set of global saddles (protocritical droplets in $\mathscr{P}$) a new stage starts that we call *supercritical*: we pass from $\mathscr{P}$ to $+\underline{1}$ through a suitable sequence of growing standard octagons with proper resistance times.

The embryonal stage is uphill; the regular, transient, and standard stages are uphill in average and, finally, the supercritical stage is downhill in average.

The first (subcritical) portions (embryonal, regular, transient, and standard) of an ε-typical path will involve notions generalizing the ones already seen in the definition of the event $\mathscr{E}_\sigma$. Embryonal, regular, transient, and standard stages in $\mathscr{E}_\sigma$ are strictly related to a particular example, with a particular choice of locations and orientations, of the corresponding ones in $\mathscr{U}_\varepsilon$. Namely, the generalization in $\mathscr{U}_\varepsilon$ with respect to $\mathscr{E}_\sigma$ is only related to geometrical transformations such as translations, rotations, or reflections with respect to some lattice axes of the corresponding clusters, octagons, and standard octagons involved in $\mathscr{E}_\sigma$. The family of sequences of 11 clusters taking part in the embryonal stage of an ε-typical path (sequentially visited in 11 steps) will be specified in great detail. Then, assigning the regular, transient, and standard stages, similarly to what we did in the definition of $\mathscr{E}_\sigma$, will consist in specifying a set of sequences of *octagons* $Q_1,..., Q_{N+1}$ with $Q_1 = Q(2)$, $Q_{N+1} = Q(D^*, D^*) \equiv$ critical nucleus, connected regions $B_1,..., B_{N+1}$ with $B_i \ni Q_i$, saddles $S_2,..., S_{N+1}$ with $S_{i+1} \in B_i \cup B_{i+1}$, and resistance times (see below).

Further, we have $N = N_r + N_t + N_s$, with N_r, N_t, and N_s given in Eq. (5.9) (the numbers of octagons in *any* sequence in the concerned stage are exactly the same as the numbers of the corresponding ones in $\mathscr{E}_\sigma$).

Then, after $Q(D^*, D^*)$ comes the supercritical stage $Q_{N+1},...,$ $Q_{N+2(M-D^*)}$, $+\underline{1}$, with the corresponding $B_{N+1},..., B_{N+2(M-D^*)}$, $S_{N+2},...,$ $S_{N+2(M-D^*)}$, and resistance times.

The regular stage and part of the transient stage will involve nonstandard octagons. Parts of the transient, standard, and supercritical stages will involve standard octagons. In the definitions introducing $\mathscr{U}_\varepsilon$, during the evolution along a typical path we are less and less specific in the following sense: at the very beginning we assign a class of sequences of clusters [no resistance times up to $Q(2)$], then, in the regular stage and part of the transient stage, we specify a class of nonstandard octagons (without specifying the sequences of nonoctagonal clusters in between). Subsequently (in part of the transient stage and in the standard stage) we will be able to specify only a class of sequences of standard octagons (without specifying the sequences of nonstandard octagons in between). Finally, in the supercritical stage, we will not even be able to specify a precise class of sequences of standard octagons and much larger fluctuations have to be allowed. We can say that, as the time goes on, our tube of trajectories becomes less and less narrow: it will correspond to the maximum possible specification compatible with an almost full probability estimate.

Now let us start with the detailed definitions.

The first part of the $\mathscr{U}_\varepsilon$, called *embryonal*, is given by the set of all paths $(\sigma_1 = \gamma_1,..., \sigma_{11} = \gamma_{11})$, where $(\gamma_1,..., \gamma_{11})$ is a generic sequence of connected clusters with (i) γ_1 given by a unit square; γ_{11} given by the saddle configuration $\bar{\gamma}_{11}$ of Scheme 5.1 arbitrarily located and oriented [the droplet $\bar{\gamma}_{11}$ can go downhill to $Q(2)$ by a single spin flip]; (ii) γ_j obtained from γ_{j-1} by adding a unit square touching it and, in this way, increasing the energy without bypassing the energy level of $\bar{\gamma}_{11}: H(\gamma_j) > H(\gamma_{j-1})$, $H(\gamma_j) < H(\bar{\gamma}_{11})$, $j = 2,..., 11$.

It is clear by inspection that, with the rule (ii), starting from γ_1, after 11 steps we always end up in γ_{11}.

An example of a sequence implementing the characteristics of the embryonal stage is given in Scheme 5.1.

Now, for the definition of the subsequent stages we have to define, preliminarily, the Q_i, B_i, and S_i.

As in $\mathscr{E}_\sigma$, for all $i = 1,..., N + 2(M - D^*)$ we take $B_i = \mathscr{B}(Q_i) =$ basin of attraction of Q_i if Q_i is a nonstandard octagon and $B_i = \mathscr{D}(D_1, D_2) =$ extended domain of attraction of Q_i if Q_i is a standard octagon. The saddles $S_i \in \partial B_i \cap B_{i+1}$, $i = 1,..., N + 2(M - D^*)$, and, for $i = 2,..., N$, $S_i \in$ {set of minimal saddles in ∂B_i}; whereas for $i = N + 1,..., N + 2(M - D^*)$, $S_i \in$ {set of minimal saddles in ∂B_{i+1}}.

Then the B_i and the S_i are determined once the Q_i are given. Any sequence $Q_1,..., Q_{N+2(M-D^*)}$ corresponding to a typical path will be called a *typical sequence* of *octagons* (it will follow the embryonal stage). The set of all sequences will be called a *typical tube* and will be denoted by $\mathscr{T}_\varepsilon$.

Now let $V_i \equiv V(Q_i)$ be given by:

(i) $V_i = h\hat{\mathscr{L}}_j$ with $\hat{\mathscr{L}}_j \equiv \min_{i=1,...,8} \mathscr{L}_i$ (see definitions before Lemma 3.5), whenever Q_i is a nonstandard octagon.

(ii) $V_i = E(L)$, whenever Q_i is a standard octagon with $\min\{L_1(Q_i), L_2(Q_i)\} = L$, with $l^* \leqslant L \leqslant L^* - 1$.

(iii) $V_i = 2J - 4K - h$, whenever Q_i is standard and supercritical, $L = \min\{L_1(Q_i), L_2(Q_i)\} \geqslant L^*$.

A path σ_t will be an *ε-typical path* if, after the embryonal stage it will visit sequentially $Q_1 \in Q(2),..., Q_{N+2(M-D^*)}$ in the following way:

Starting from Q_i, it will spend some time t_i inside B_i.

Then, after passing through S_{i+1}, it will reach Q_{i+1} for the first time. Calling t_i the time interval between first arrivals in Q_i and Q_{i+1}, we have

$$\exp\{\beta(V_i - \varepsilon)\} < t_i < \exp\{\beta(V_i + \varepsilon)\}$$

So, to conclude the definition of $\mathscr{U}_\varepsilon$, we only need to assign the typical tube of octagons. We start this definition by distinguishing the regular, transient, standard, and supercritical portions.

The *regular portion* of the typical tube is denoted by $\mathscr{T}_\varepsilon^{(r)}$. Starting from $Q(2)$, $\mathscr{T}_\varepsilon^{(r)}$ contains the set of all the reverse of a sequence of canonical contractions (see Section 3) from $Q(3)$ to $Q(2)$ and so on up to $Q(l^*)$. The transient and standard portions $\mathscr{T}_\varepsilon^{(t)}$ and $\mathscr{T}_\varepsilon^{(s)}$ of the typical tube are simply given by the set of sequences of octagons obtained by the set of sequences appearing in the definitions given above when introducing $\mathscr{E}_\sigma$, modulo translations and the obvious rotations and reflections (we do not enter into a detailed classification, leaving the easy exercise to the interested reader).

After getting $S_{N+1} \in \mathscr{P}$ the supercritical stage $\mathscr{T}_\varepsilon^{(sc)}$ starts.

It will consist of the set of sequences of standard octagons

$Q(D_1^{(j)}, D_2^{(j)})$, $j = 1, \ldots, 2(M - D^*)$, with the following monotonicity property:

$$\text{either} \quad (D_1^{(j+1)}, D_2^{(j+1)}) = (D_1^{(j)} + 1, D_2^{(j)})$$
$$\text{or} \quad (D_1^{(j+1)}, D_2^{(j+1)}) = (D_1^{(j)}, D_2^{(j)} + 1)$$

Proof of Theorem 3. Theorem 3 follows directly by Propositions 1–4 and the easy observation that the embryonal path is the reverse of a downhill shrinking starting from γ_{11}, via a straightforward adaptation of the results of ref. 14 based on reversibility of the process (Lemmas 2–4 and Theorem 1 therein).

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